

General Solutions of Relativistic Wave Equations II: Arbitrary Spin Chains

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A construction of relativistic wave equations on the homogeneous spaces of the Poincaré group is given for arbitrary spin chains. Parametrizations of the field functions and harmonic analysis on the homogeneous spaces are studied. It is shown that a direct product of Minkowski space time and two-dimensional complex sphere is the most suitable homogeneous space for the physical applications. The Lagrangian formalism and field equations on the Poincaré and Lorentz groups are considered. A boundary value problem for the relativistically invariant system is defined. General solutions of this problem are expressed via an expansion in hyperspherical functions defined on the complex two-sphere.

KEY WORDS: relativistic wave equations; fields on the Poincaré group; harmonic analysis; boundary value problem.

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1. INTRODUCTION

The theory of relativistic wave equations (RWE) is one of the oldest topics in theoretical physics. As usual, the theory of RWE closely relates with higher spin formalisms. However, at present, there is no a fully adequate formalism for description of higher spin fields – all widely accepted higher spin formalisms such as Rarita–Schwinger approach (Rarita and Schwinger, 1941; Bargmann and Wigner, 1948; Gel’fand-Yaglom, 1948) multi-spinor theories, and also Joos–Weinberg $2(2j + 1)$ -component formalism (Joos, 1962; Weinberg, 1969) have many intrinsic contradictions and difficulties. On the other hand, it is known that the study of RWE leads naturally to the fields which depend on both Minkowski space coordinates and some continuous variables corresponding to spin degrees of freedom (Ginzburg and Tamm, 1947; Bargmann and Wigner, 1948; Yukawa, 1950; Shirokov, 1951) Wave functions of this type can be treated as the fields

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on homogeneous spaces of the Poincaré group. These fields were first studied by Finkelstein (1955), he gave a classification and explicit constructions of homogeneous spaces of the Poincaré group. The general form of these fields closely relates with the structure of the Lorentz and Poincaré group representations (Gel'fand *et al.*, 1963; Naimark, 1964; Biedenharn, 1988; Gitman and Shelepin, 2001) and admits the following factorization $f(x, \mathbf{z}) = \phi^n(\mathbf{z})\psi_n(x)$, where $x \in T_4$ and $\phi^n(\mathbf{z})$ form a basis in the representation space of the Lorentz group, T_4 is the group of four-dimensional translations. In addition to a general theory of relativistic wave equations, a search of solutions for RWE has a great importance (Bagrov and Gitman, 1989).

In the present work, we study solutions of RWE in terms of the fields defined on the two-dimensional complex sphere. This sphere is a homogeneous space of the Lorentz group (in this case, the field functions $f(x, \mathbf{z})$ reduce to $\phi^n(\mathbf{z})$). In the previous work (Varlamov, 2003), solutions of RWE were obtained in the form of expansions in hyperspherical functions for the representations of the type $(l, 0) \oplus (0, \dot{l})$. Solutions (wavefunctions on the group manifold \mathcal{M}_{10}) of the simplest RWE (Dirac $(1/2, 0) \oplus (0, 1/2)$ and Maxwell $(1, 0) \oplus (0, 1)$ fields) have been given in Varlamov (2003, 2004). In this work, we study more general representations $(l_1, l_2) \oplus (l_2, l_1)$ (tensor representations of the proper orthochronous Lorentz group \mathfrak{G}_+) which correspond to arbitrary spin chains. In turn, arbitrary spin chains contain interlocking representations of the group \mathfrak{G}_+ as a particular case. It is known that the higher spin fields and, correspondingly, composite elementary particles can be formulated in terms of interlocking representations of \mathfrak{G}_+ (Bhabha–Gel'fand–Yaglom chains). The stable composite particle corresponds to an indecomposable equation defined within interlocking representation, and vice versa, unstable particles are described by decomposable equations.

The present paper is organized as follows. In the Section 2, we consider some basic facts concerning the group $SL(2, \mathbb{C})$. The main object of this section is a local isomorphism $SL(2, \mathbb{C}) \simeq SU(2) \otimes SU(2)$. We introduce here hyperspherical functions for the tensor representations and further we study recurrence relations between these functions. It is shown that matrix elements of the representations $(l_1, \dot{l}_2) \oplus (l_2, \dot{l}_1)$ are expressed via the hyperspherical functions. In the Section 3, we give a brief introduction to the fields on the Poincaré group \mathcal{P} . We consider fields on the group manifold \mathcal{M}_{10} and on the eight-dimensional homogeneous space $\mathcal{M}_8 = \mathbb{R}^{1,3} \times \mathbb{S}^2$ of \mathcal{P} , where $\mathbb{R}^{1,3}$ is the Minkowski spacetime and \mathbb{S}^2 is the two-dimensional complex sphere. In parallel, we consider basic facts concerning harmonic analysis on the homogeneous spaces \mathcal{M}_{10} , \mathcal{M}_8 , and also on the group $SL(2, \mathbb{C})$ and on the sphere \mathbb{S}^2 . The following logical step consists in definition of the Lagrangian formalism and field equations on the homogeneous spaces of \mathcal{P} . The field equations for arbitrary spin are derived in the Section 4, by the standard variation procedure from a selected Lagrangian. The explicit construction of RWE for the arbitrary spin chains of \mathfrak{G}_+ is given in the Section 5. In the Section 6, we

separate variables in a relativistically invariant system via the recurrence relations between hyperspherical functions. First of all, we set up a boundary value problem for RWE, which can be treated as a Dirichlet problem for a complex ball. It is shown that solutions of this problem are expressed via Fourier type series on the two-dimensional complex sphere.

2. THE GROUP $SL(2, \mathbb{C})$

As is known, the group $SL(2, \mathbb{C})$ is an universal covering of the proper orthochronous Lorentz group \mathfrak{G}_+ . The group $SL(2, \mathbb{C})$ of all complex matrices

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of 2nd order with the determinant $\alpha\delta - \gamma\beta = 1$, is a *complexification* of the group $SU(2)$. The group $SU(2)$ is one of the real forms of $SL(2, \mathbb{C})$. The transition from $SU(2)$ to $SL(2, \mathbb{C})$ is realized via the complexification of three real parameters φ, θ, ψ (Euler angles). Let $\theta^c = \theta - i\tau, \varphi^c = \varphi - i\epsilon, \psi^c = \psi - i\varepsilon$ be complex Euler angles, where

$$\begin{aligned} 0 \leq \operatorname{Re} \theta^c = \theta &\leq \pi, & -\infty < \operatorname{Im} \theta^c = \tau < +\infty, \\ 0 \leq \operatorname{Re} \varphi^c = \varphi &< 2\pi, & -\infty < \operatorname{Im} \varphi^c = \epsilon < +\infty, \\ -2\pi \leq \operatorname{Re} \psi^c = \psi &< 2\pi, & -\infty < \operatorname{Im} \psi^c = \varepsilon < +\infty. \end{aligned} \tag{1}$$

The group $SL(2, \mathbb{C})$ has six one-parameter subgroups

$$a_1(t) = \begin{pmatrix} \cos \frac{t}{2} & i \sin \frac{t}{2} \\ i \sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix}, \quad a_2(t) = \begin{pmatrix} \cos \frac{t}{2} & -\sin \frac{t}{2} \\ \sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix}, \quad a_3(t) = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix},$$

$$b_1(t) = \begin{pmatrix} \cos h \frac{t}{2} & \sin h \frac{t}{2} \\ \sin h \frac{t}{2} & \cos h \frac{t}{2} \end{pmatrix}, \quad b_2(t) = \begin{pmatrix} \cos h \frac{t}{2} & i \sin h \frac{t}{2} \\ -i \sin h \frac{t}{2} & \cos h \frac{t}{2} \end{pmatrix},$$

$$b_3(t) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

The tangent matrices A_i and B_i of these subgroups are defined as follows

$$\begin{aligned} A_1 &= \left. \frac{da_1(t)}{dt} \right|_{t=0} = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ A_2 &= \left. \frac{da_2(t)}{dt} \right|_{t=0} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
 A_3 &= \left. \frac{da_3(t)}{dt} \right|_{t=0} = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
 B_1 &= \left. \frac{db_1(t)}{dt} \right|_{t=0} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
 B_2 &= \left. \frac{db_2(t)}{dt} \right|_{t=0} = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
 B_3 &= \left. \frac{db_3(t)}{dt} \right|_{t=0} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
 \end{aligned}$$

The elements A_i and B_i form a basis of Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and satisfy the relations

$$\left. \begin{aligned}
 [A_1, A_2] &= A_3, & [A_2, A_3] &= A_1, & [A_3, A_1] &= A_2, \\
 [B_1, B_2] &= -A_3, & [B_2, B_3] &= -A_1, & [B_3, B_1] &= -A_2, \\
 [A_1, B_1] &= 0, & [A_2, B_2] &= 0, & [A_3, B_3] &= 0, \\
 [A_1, B_2] &= B_3, & [A_1, B_3] &= -B_2, \\
 [A_2, B_3] &= B_1, & [A_2, B_1] &= -B_3, \\
 [A_3, B_1] &= B_2, & [A_3, B_2] &= -B_1.
 \end{aligned} \right\} \quad (2)$$

Let us consider the operators

$$\begin{aligned}
 X_l &= \frac{1}{2}i(A_l + iB_l), & Y_l &= \frac{1}{2}i(A_l - iB_l) \\
 (l &= 1, 2, 3)
 \end{aligned} \quad (3)$$

Using the relations (2), we find that

$$[X_k, X_l] = i\varepsilon_{klm}X_m, \quad [Y_l, Y_m] = i\varepsilon_{lmn}Y_n, \quad [X_l, Y_m] = 0. \quad (4)$$

Further, introducing generators of the form

$$\left. \begin{aligned}
 X_+ &= X_1 + iX_2, & X_- &= X_1 - iX_2, \\
 Y_+ &= Y_1 + iY_2, & Y_- &= Y_1 - iY_2,
 \end{aligned} \right\} \quad (5)$$

we see that in virtue of commutativity of the relations (4) a space of an irreducible finite-dimensional representation of the group $SL(2, \mathbb{C})$ can be spanned on the totality of $(2l+1)(2\dot{l}+1)$ basis vectors $|l, m; \dot{l}, \dot{m}\rangle$, where l, m, \dot{l}, \dot{m} are integer or half-integer numbers, $-l \leq m \leq l$, $-\dot{l} \leq \dot{m} \leq \dot{l}$. Therefore,

$$\begin{aligned}
 X_- |l, m; \dot{l}, \dot{m}\rangle &= \sqrt{(l+m)(l-m+1)} |l, m-1; \dot{l}, \dot{m}\rangle \quad (m > -l), \\
 X_+ |l, m; \dot{l}, \dot{m}\rangle &= \sqrt{(l-m)(l+m+1)} |l, m+1; \dot{l}, \dot{m}\rangle \quad (m < l), \\
 X_3 |l, m; \dot{l}, \dot{m}\rangle &= m |l, m; \dot{l}, \dot{m}\rangle,
 \end{aligned}$$

$$\begin{aligned}
 Y_- |l, m; \dot{l}, \dot{m}\rangle &= \sqrt{(\dot{l} + \dot{m})(\dot{l} - \dot{m} + 1)} |l, m; \dot{l}, \dot{m} - 1\rangle \quad (\dot{m} > -\dot{l}), \\
 Y_+ |l, m; \dot{l}, \dot{m}\rangle &= \sqrt{(\dot{l} - \dot{m})(\dot{l} + \dot{m} + 1)} |l, m; \dot{l}, \dot{m} + 1\rangle \quad (\dot{m} < \dot{l}), \\
 Y_3 |l, m; \dot{l}, \dot{m}\rangle &= \dot{m} |l, m; \dot{l}, \dot{m}\rangle.
 \end{aligned}
 \tag{6}$$

From the relations, (4) it follows that each of the sets of infinitesimal operators X and Y generates the group $SU(2)$ and these two groups commute with each other. Thus, from the relations (4) and (6) it follows that the group $SL(2, \mathbb{C})$, in essence, is equivalent locally to the group $SU(2) \otimes SU(2)$. It should be noted here that the representation basis, defined by the formulae (3)–(6), has an evident physical meaning. For example, in the case of $(1, 0) \oplus (0, 1)$ representation space there is an analogy with the photon spin states. Namely, the operators X and Y correspond to the right and left polarization states of the photon. For that reason, we will call the canonical basis consisting of the vectors $|lm; \dot{l}\dot{m}\rangle$ as a *helicity basis*.

In the work conducted by Varlamov (2003), we studied RWE for the fields $(l, 0) \oplus (0, \dot{l})$ which described within the representation $\tau_{l,0} \oplus \tau_{0,\dot{l}}$ of the group \mathfrak{G}_+ . In this work, we study more general representations of the type $\tau_{l,\dot{l}} \oplus \tau_{\dot{l},l}$ (tensor representations), where $\tau_{l,\dot{l}} = \tau_{l,0} \otimes \tau_{0,\dot{l}}$ and $\tau_{\dot{l},l} = \tau_{\dot{l},0} \otimes \tau_{0,l}$. It is obvious that these representations include $\tau_{l,0} \oplus \tau_{0,\dot{l}}$ as a particular case. In general, a tensor structure of the infinitesimal operators has the form

$$\begin{aligned}
 A_i^{ll} &= A_i^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes A_i^l, \\
 B_i^{ll} &= B_i^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes B_i^l, \\
 \tilde{A}_i^{ll} &= A_i^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes A_i^l, \\
 \tilde{B}_i^{ll} &= B_i^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes B_i^l,
 \end{aligned}
 \tag{7}$$

where A_i^l, B_i^l and $A_i^{\dot{l}}, B_i^{\dot{l}}$ are infinitesimal operators of the representations $\tau_{l,0}$ and $\tau_{0,\dot{l}}$, respectively ($i = 1, 2, 3$) Or, more explicitly,

$$\begin{aligned}
 A_1^{ll} |l, m; \dot{l}, \dot{m}\rangle &= -\frac{i}{2} \alpha_m^l |l, m - 1; \dot{l}, \dot{m}\rangle - \frac{i}{2} \alpha_{m+1}^l |l, m + 1; \dot{l}, \dot{m}\rangle \\
 &+ \frac{i}{2} \alpha_{\dot{m}}^{\dot{l}} |l, m; \dot{l}, \dot{m} - 1\rangle + \frac{i}{2} \alpha_{\dot{m}+1}^{\dot{l}} |l, m; \dot{l}, \dot{m} + 1\rangle,
 \end{aligned}
 \tag{8}$$

$$\begin{aligned}
 A_2^{ll} |l, m; \dot{l}, \dot{m}\rangle &= \frac{1}{2} \alpha_m^l |l, m - 1; \dot{l}, \dot{m}\rangle - \frac{1}{2} \alpha_{m+1}^l |l, m + 1; \dot{l}, \dot{m}\rangle \\
 &- \frac{1}{2} \alpha_{\dot{m}}^{\dot{l}} |l, m; \dot{l}, \dot{m} - 1\rangle + \frac{1}{2} \alpha_{\dot{m}+1}^{\dot{l}} |l, m; \dot{l}, \dot{m} + 1\rangle,
 \end{aligned}
 \tag{9}$$

$$A_3^{ll} |l, m; \dot{l}, \dot{m}\rangle = -im |l, m; \dot{l}, \dot{m}\rangle + i\dot{m} |l, m; \dot{l}, \dot{m}\rangle,
 \tag{10}$$

$$\begin{aligned} B_1^{ll} | l, m; \dot{l}, \dot{m} \rangle &= -\frac{1}{2} \alpha_m^l | l, m-1; \dot{l}, \dot{m} \rangle - \frac{1}{2} \alpha_{m+1}^l | l, m+1; \dot{l}, \dot{m} \rangle \\ &\quad - \frac{1}{2} \alpha_{\dot{m}}^l | l, m; \dot{l}, \dot{m}-1 \rangle - \frac{1}{2} \alpha_{\dot{m}+1}^l | l, m; \dot{l}, \dot{m}+1 \rangle, \end{aligned} \quad (11)$$

$$\begin{aligned} B_2^{ll} | l, m; \dot{l}, \dot{m} \rangle &= -\frac{i}{2} \alpha_m^l | l, m-1; \dot{l}, \dot{m} \rangle + \frac{i}{2} \alpha_{m+1}^l | l, m+1; \dot{l}, \dot{m} \rangle \\ &\quad - \frac{i}{2} \alpha_{\dot{m}}^l | l, m; \dot{l}, \dot{m}-1 \rangle + \frac{i}{2} \alpha_{\dot{m}+1}^l | l, m; \dot{l}, \dot{m}+1 \rangle, \end{aligned} \quad (12)$$

$$B_3^{ll} | l, m; \dot{l}, \dot{m} \rangle = -m | l, m; \dot{l}, \dot{m} \rangle - \dot{m} | l, m; \dot{l}, \dot{m} \rangle, \quad (13)$$

$$\begin{aligned} \tilde{A}_1^{ll} | l, m; \dot{l}, \dot{m} \rangle &= -\frac{i}{2} \alpha_{\dot{m}}^l | l, m; \dot{l}, \dot{m}-1 \rangle - \frac{i}{2} \alpha_{\dot{m}+1}^l | l, m; \dot{l}, \dot{m}+1 \rangle \\ &\quad + \frac{i}{2} \alpha_m^l | l, m-1; \dot{l}, \dot{m} \rangle + \frac{i}{2} \alpha_{m+1}^l | l, m+1; \dot{l}, \dot{m} \rangle, \end{aligned} \quad (14)$$

$$\begin{aligned} \tilde{A}_2^{ll} | l, m; \dot{l}, \dot{m} \rangle &= \frac{1}{2} \alpha_{\dot{m}}^l | l, m; \dot{l}, \dot{m}-1 \rangle - \frac{1}{2} \alpha_{\dot{m}+1}^l | l, m; \dot{l}, \dot{m}+1 \rangle \\ &\quad + \frac{1}{2} \alpha_m^l | l, m-1; \dot{l}, \dot{m} \rangle + \frac{1}{2} \alpha_{m+1}^l | l, m+1; \dot{l}, \dot{m} \rangle, \end{aligned} \quad (15)$$

$$\tilde{A}_3^{ll} | l, m; \dot{l}, \dot{m} \rangle = -i\dot{m} | l, m; \dot{l}, \dot{m} \rangle + im | l, m; \dot{l}, \dot{m} \rangle, \quad (16)$$

$$\begin{aligned} \tilde{B}_1^{ll} | l, m; \dot{l}, \dot{m} \rangle &= \frac{1}{2} \alpha_{\dot{m}}^l | l, m; \dot{l}, \dot{m}-1 \rangle + \frac{1}{2} \alpha_{\dot{m}+1}^l | l, m; \dot{l}, \dot{m}+1 \rangle \\ &\quad + \frac{1}{2} \alpha_m^l | l, m-1; \dot{l}, \dot{m} \rangle + \frac{1}{2} \alpha_{m+1}^l | l, m+1; \dot{l}, \dot{m} \rangle, \end{aligned} \quad (17)$$

$$\begin{aligned} \tilde{B}_2^{ll} | l, m; \dot{l}, \dot{m} \rangle &= \frac{i}{2} \alpha_{\dot{m}}^l | l, m; \dot{l}, \dot{m}-1 \rangle - \frac{i}{2} \alpha_{\dot{m}+1}^l | l, m; \dot{l}, \dot{m}+1 \rangle \\ &\quad + \frac{i}{2} \alpha_m^l | l, m-1; \dot{l}, \dot{m} \rangle - \frac{i}{2} \alpha_{m+1}^l | l, m+1; \dot{l}, \dot{m} \rangle, \end{aligned} \quad (18)$$

$$\tilde{B}_3^{ll} | l, m; \dot{l}, \dot{m} \rangle = \dot{m} | l, m; \dot{l}, \dot{m} \rangle + m | l, m; \dot{l}, \dot{m} \rangle. \quad (19)$$

Further, let us define generators X_-^{ll} , X_+^{ll} , Y_-^{ll} , Y_+^{ll} and \tilde{X}_-^{ll} , \tilde{X}_+^{ll} , \tilde{Y}_-^{ll} , \tilde{Y}_+^{ll} following the rule (5). It is easy to verify that these generators satisfy the relations of the type (6).

On the group $SL(2, \mathbb{C})$, there exist the following Laplace–Beltrami operators:

$$\begin{aligned} X^2 &= X_1^2 + X_2^2 + X_3^2 = \frac{1}{4}(A^2 - B^2 + 2iAB), \\ Y^2 &= Y_1^2 + Y_2^2 + Y_3^2 = \frac{1}{4}(\tilde{A}^2 - \tilde{B}^2 - 2i\tilde{A}\tilde{B}) \end{aligned} \quad (20)$$

At this point, we see that operators (20) contain the well known Casimir operators $A^2 - B^2$, AB of the Lorentz group. Using expressions (1), we obtain an Euler parametrization of the Laplace–Beltrami operators,

$$\begin{aligned} X^2 &= \frac{\partial^2}{\partial \theta^{c2}} + \cot \theta^c \frac{\partial}{\partial \theta^c} + \frac{1}{\sin^2 \theta^c} \left[\frac{\partial^2}{\partial \varphi^{c2}} - 2 \cos \theta^c \frac{\partial}{\partial \varphi^c} \frac{\partial}{\partial \psi^c} + \frac{\partial^2}{\partial \psi^{c2}} \right], \\ Y^2 &= \frac{\partial^2}{\partial \hat{\theta}^{c2}} + \cot \hat{\theta}^c \frac{\partial}{\partial \hat{\theta}^c} + \frac{1}{\sin^2 \hat{\theta}^c} \left[\frac{\partial^2}{\partial \hat{\varphi}^{c2}} - 2 \cos \hat{\theta}^c \frac{\partial}{\partial \hat{\varphi}^c} \frac{\partial}{\partial \hat{\psi}^c} + \frac{\partial^2}{\partial \hat{\psi}^{c2}} \right]. \end{aligned} \quad (21)$$

Matrix elements $t_{mn;\hat{m}\hat{n}}^{ll}(\mathbf{g}) = \mathfrak{M}_{mn;\hat{m}\hat{n}}^{ll}(\varphi^c, \theta^c, \psi^c)$ of irreducible representations of $SL(2, \mathbb{C})$ are eigenfunctions of the operators (21),

$$\begin{aligned} [X^2 + l(l+1)] \mathfrak{M}_{mn;\hat{m}\hat{n}}^{ll}(\varphi^c, \theta^c, \psi^c) &= 0, \\ [Y^2 + l(l+1)] \mathfrak{M}_{mn;\hat{m}\hat{n}}^{ll}(\hat{\varphi}^c, \hat{\theta}^c, \hat{\psi}^c) &= 0, \end{aligned} \quad (22)$$

where

$$\mathfrak{M}_{mn;\hat{m}\hat{n}}^{ll}(\mathbf{g}) = e^{-i(m\varphi^c + n\psi^c)} \mathfrak{Z}_{mn;\hat{m}\hat{n}}^{ll}(\cos \theta^c, \cos \hat{\theta}^c) e^{i(\hat{m}\hat{\varphi}^c + \hat{n}\hat{\psi}^c)}. \quad (23)$$

Substituting the functions (23) into (22) and taking into account the operators (21) and substitutions $z = \cos \theta^c$, $z^* = \cos \hat{\theta}^c$, we come to the following differential equations (a complex analog of the Legendre equations):

$$\left[(1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{m^2 + n^2 - 2mnz}{1 - z^2} + l(l+1) \right] \mathfrak{Z}_{mn;\hat{m}\hat{n}}^{ll}(z, z^*) = 0, \quad (24)$$

$$\left[(1 - z^{*2}) \frac{d^2}{dz^{*2}} - 2z^* \frac{d}{dz^*} - \frac{\hat{m}^2 + \hat{n}^2 - 2\hat{m}\hat{n}z^*}{1 - z^{*2}} + l(l+1) \right] \mathfrak{Z}_{mn;\hat{m}\hat{n}}^{ll}(z, z^*) = 0. \quad (25)$$

The latter equations have three singular points $-1, +1, \infty$. Solutions of (24) and (25) have the form

$$\mathfrak{Z}_{mn;\hat{m}\hat{n}}^{ll}(\cos \theta^c, \cos \hat{\theta}^c) = Z_{mn}^l(\cos \theta^c) Z_{\hat{m}\hat{n}}^l(\cos \hat{\theta}^c), \quad (26)$$

where

$$\begin{aligned} Z_{mn}^l(\cos \theta^c) &= \sum_{k=-l}^l i^{m-k} \sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)} \\ &\quad \times \cos^{2l} \frac{\theta}{2} \tan^{m-k} \frac{\theta}{2} \\ &\quad \times \sum_{j=\max(0, k-m)}^{\min(l-m, l+k)} \frac{i^{2j} \tan^{2j} \frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)} \end{aligned}$$

$$\begin{aligned} & \times \sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-k+1)\Gamma(l+k+1)} \cos h^{2l} \frac{\tau}{2} \tan h^{n-k} \frac{\tau}{2} \\ & \times \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \frac{\tan h^{2s} \frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)}. \end{aligned} \quad (27)$$

and $Z_{\bar{m}\bar{n}}^l$ is a complex conjugate function with respect to Z_{mn}^l . We will call the functions $\mathfrak{Z}_{mn;\bar{m}\bar{n}}^l$ in (26) as *tensor hyperspherical functions*.

Using the formula (26), we find explicit expressions for the matrices $T_{ll}(\theta, \tau)$ of the finite-dimensional representations $\tau_{\frac{1}{2}\frac{1}{2}}$ and $\tau_{1\frac{1}{2}}$:

$$\begin{aligned} \tau_{\frac{1}{2}\frac{1}{2}}(\theta, \tau) &= \begin{pmatrix} Z_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} & Z_{\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} \\ Z_{-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} & Z_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} \end{pmatrix} \otimes \begin{pmatrix} *_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} & *_{\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} \\ *_{-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} & *_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{Z}_{-\frac{1}{2}\frac{1}{2};-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{\frac{1}{2}\frac{1}{2};-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} \\ \mathfrak{Z}_{-\frac{1}{2}\frac{1}{2};-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{\frac{1}{2}\frac{1}{2};-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} \\ \mathfrak{Z}_{-\frac{1}{2}\frac{1}{2};-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{\frac{1}{2}\frac{1}{2};-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} \\ \mathfrak{Z}_{-\frac{1}{2}\frac{1}{2};-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{\frac{1}{2}\frac{1}{2};-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} & \mathfrak{Z}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}\frac{1}{2}} \end{pmatrix}, \end{aligned} \quad (28)$$

$$\begin{aligned} \tau_{1\frac{1}{2}}(\theta, \tau) &= \begin{pmatrix} Z_{-1-1}^1 & Z_{-10}^1 & Z_{-11}^1 \\ Z_{0-1}^1 & Z_{00}^1 & Z_{01}^1 \\ Z_{1-1}^1 & Z_{10}^1 & Z_{11}^1 \end{pmatrix} \otimes \begin{pmatrix} *_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} & *_{\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} \\ *_{-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} & *_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} \\ Z_{-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} & Z_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{Z}_{-1-1;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-1-1;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-10;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-10;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-11;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-11;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} \\ \mathfrak{Z}_{-1-1;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-1-1;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-10;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-10;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-11;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{-11;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} \\ \mathfrak{Z}_{0-1;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{0-1;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{00;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{00;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{01;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{01;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} \\ \mathfrak{Z}_{0-1;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{0-1;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{00;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{00;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{01;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{01;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} \\ \mathfrak{Z}_{1-1;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{1-1;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{10;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{10;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{11;-\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{11;\frac{1}{2}-\frac{1}{2}}^{1\frac{1}{2}} \\ \mathfrak{Z}_{1-1;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{1-1;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{10;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{10;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{11;-\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} & \mathfrak{Z}_{11;\frac{1}{2}\frac{1}{2}}^{1\frac{1}{2}} \end{pmatrix}. \end{aligned} \quad (29)$$

2.1. Recurrence Relations Between the Functions $\mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}(\cos \theta^c, \cos \dot{\theta}^c)$

Between the hyperspherical functions $\mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}$ there exists a wide variety of recurrence relations. The part of them relates the hyperspherical functions of one and the same order (with identical l and \dot{l}), other part relates the functions of different orders.

In virtue of the representation (6), the recurrence formulae for the hyperspherical functions of one and the same order follow from the equalities

$$X_-^{ll} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll} = \alpha_n \mathfrak{M}_{m,n-1;\dot{m}\dot{n}}^{ll}, \quad X_+^{ll} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll} = \alpha_{n+1} \mathfrak{M}_{m,n+1;\dot{m}\dot{n}}^{ll}, \quad (30)$$

$$Y_-^{ll} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll} = \alpha_{\dot{n}} \mathfrak{M}_{mn;\dot{m}\dot{n}-1}^{ll}, \quad Y_+^{ll} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll} = \alpha_{\dot{n}+1} \mathfrak{M}_{mn;\dot{m}\dot{n}+1}^{ll}. \quad (31)$$

As it is shown in Varlamov (2003), the generators (5) can be expressed via the complex Euler angles as

$$X_+ = \frac{e^{-i\psi^c}}{2} \left[i \frac{\partial}{\partial \theta} - \frac{1}{\sin \theta^c} \frac{\partial}{\partial \varphi} + \cot \theta^c \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \tau} - \frac{i}{\sin \theta^c} \frac{\partial}{\partial \epsilon} + i \cot \theta^c \frac{\partial}{\partial \varepsilon} \right], \quad (32)$$

$$X_- = \frac{e^{i\psi^c}}{2} \left[i \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta^c} \frac{\partial}{\partial \varphi} - \cot \theta^c \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \tau} + \frac{i}{\sin \theta^c} \frac{\partial}{\partial \epsilon} - i \cot \theta^c \frac{\partial}{\partial \varepsilon} \right], \quad (33)$$

$$Y_+ = \frac{e^{-i\dot{\psi}^c}}{2} \left[i \frac{\partial}{\partial \theta} - \frac{1}{\sin \dot{\theta}^c} \frac{\partial}{\partial \varphi} + \cot \dot{\theta}^c \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \tau} + \frac{i}{\sin \dot{\theta}^c} \frac{\partial}{\partial \epsilon} - i \cot \dot{\theta}^c \frac{\partial}{\partial \varepsilon} \right], \quad (34)$$

$$Y_- = \frac{e^{i\dot{\psi}^c}}{2} \left[i \frac{\partial}{\partial \theta} + \frac{1}{\sin \dot{\theta}^c} \frac{\partial}{\partial \varphi} - \cot \dot{\theta}^c \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \tau} - \frac{i}{\sin \dot{\theta}^c} \frac{\partial}{\partial \epsilon} + i \cot \dot{\theta}^c \frac{\partial}{\partial \varepsilon} \right]. \quad (35)$$

Substituting the function $\mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll} = e^{-m(\epsilon+i\varphi)-n(\varepsilon+i\psi)} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}(\theta, \tau) e^{-\dot{m}(\epsilon-i\varpi)-\dot{n}(\varepsilon-i\psi)}$ into the relations (30) and taking into account the operators (32) and (33), we find that

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \theta} - \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \tau} - \frac{2i(m-n \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll} = 2\alpha_n \mathfrak{Z}_{m,n-1;\dot{m}\dot{n}}^{ll},$$

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \theta} - \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \tau} + \frac{2i(m-n \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll} = 2\alpha_{n+1} \mathfrak{Z}_{m,n+1;\dot{m}\dot{n}}^{ll}.$$

Since the functions $\mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}$ are symmetric, that is, $\mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll} = \mathfrak{Z}_{\dot{m}\dot{n};mn}^{ll}$, then substituting $\mathfrak{Z}_{nm;\dot{m}\dot{n}}^{ll}$ instead of $\mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}$ and replacing m by n , and n by m , we obtain

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \theta} - \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \tau} - \frac{2i(n - m \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll} = 2\alpha_m \mathfrak{Z}_{m-1,n;\dot{m}\dot{n}}^{ll}, \quad (36)$$

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \theta} - \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \tau} + \frac{2i(n - m \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll} = 2\alpha_{m+1} \mathfrak{Z}_{m+1,n;\dot{m}\dot{n}}^{ll}. \quad (37)$$

Analogously, from the relations (31) and generators (34) and (35), we have

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \theta} + \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \tau} + \frac{2i(\dot{n} - \dot{m} \cos \dot{\theta}^c)}{\sin \dot{\theta}^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll} = 2\alpha_{\dot{m}} \mathfrak{Z}_{mn;\dot{m}-1,\dot{n}}^{ll}, \quad (38)$$

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \theta} + \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}}{\partial \tau} - \frac{2i(\dot{n} - \dot{m} \cos \dot{\theta}^c)}{\sin \dot{\theta}^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll} = 2\alpha_{\dot{m}+1} \mathfrak{Z}_{mn;\dot{m}+1,\dot{n}}^{ll}. \quad (39)$$

Further, for the conjugate representations we have

$$\tilde{\mathcal{X}}_-^{ll} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{*ll} = \alpha_n \mathfrak{M}_{m,n-1;\dot{m}\dot{n}}^{*ll}, \quad \tilde{\mathcal{X}}_+^{ll} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{*ll} = \alpha_{n+1} \mathfrak{M}_{m,n+1;\dot{m}\dot{n}}^{*ll}, \quad (40)$$

$$\tilde{\mathcal{Y}}_-^{ll} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{*ll} = \alpha_{\dot{n}} \mathfrak{M}_{m,n-1;\dot{m}\dot{n}}^{*ll}, \quad \tilde{\mathcal{Y}}_+^{ll} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{*ll} = \alpha_{\dot{n}+1} \mathfrak{M}_{m,n+1;\dot{m}\dot{n}}^{*ll}, \quad (41)$$

where $\tilde{\mathcal{X}}_-^{ll} \sim \mathcal{Y}_-^{ll}$, $\tilde{\mathcal{X}}_+^{ll} \sim \mathcal{Y}_+^{ll}$, $\tilde{\mathcal{Y}}_-^{ll} \sim \mathcal{X}_-^{ll}$, $\tilde{\mathcal{Y}}_+^{ll} \sim \mathcal{X}_+^{ll}$. Substituting now the function $\mathfrak{M}_{mn;\dot{m}\dot{n}}^{*ll} = e^{-m(\epsilon-i\varphi)-n(\epsilon-i\psi)} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}(\theta, \tau) e^{-\dot{m}(\epsilon+i\varphi)-\dot{n}(\epsilon+i\psi)}$ into the relations (40) and (41), we obtain

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}}{\partial \theta} + \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}}{\partial \tau} + \frac{2i(n - m \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll} = 2\alpha_m \mathfrak{Z}_{m-1,n;\dot{m}\dot{n}}^{*ll}, \quad (42)$$

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}}{\partial \theta} + \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}}{\partial \tau} - \frac{2i(n - m \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll} = 2\alpha_{m+1} \mathfrak{Z}_{m+1,n;\dot{m}\dot{n}}^{*ll}. \quad (43)$$

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}}{\partial \theta} - \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}}{\partial \tau} - \frac{2i(\dot{n} - \dot{m} \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll} = 2\alpha_{\dot{m}} \mathfrak{Z}_{mn;\dot{m}-1,\dot{n}}^{*ll}, \quad (44)$$

$$i \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}}{\partial \theta} - \frac{\partial \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll}}{\partial \tau} + \frac{2i(\dot{n} - \dot{m} \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{*ll} = 2\alpha_{\dot{m}+1} \mathfrak{Z}_{mn;\dot{m}+1,\dot{n}}^{*ll}. \quad (45)$$

3. FIELDS ON THE POINCARÉ GROUP

Fields on the Poincaré group present itself a natural generalization of the concept of wave function. These fields (generalized wave functions) were introduced independently by several authors (Ginzburg, Tamm, 1947; Bargmann and Wigner, 1948; Yukawa, 1950; Shirokov, 1951) mainly in connection with constructing relativistic wave equations. The following logical step was done by Finkelstein (1955), he suggested to consider the wave function depending both the coordinates on the Minkowski spacetime and some continuous variables corresponding to spin degrees of freedom (internal space). In essence, this generalization consists in replacing the Minkowski space by a larger space on which the Poincaré group acts. If this action is to be transitive, one is lead to consider the homogeneous spaces of the Poincaré group. All the homogeneous spaces of this type were listed by Finkelstein (1955) and by Bacry and Kihlberg (1969) and the fields on these spaces were considered in the works conducted by (Lurçat, 1964; Bacry, 1967; Nilsson and Beskow, 1967; Kihlberg, 1968; Kihlberg, 1970; Toller, 1996; Gitman and Shelepin, 2001).

A homogeneous space \mathcal{M} of a group G has the following properties:

- (a) It is a topological space on which the group G acts continuously, that is, let y be a point in \mathcal{M} , then gy is defined and is again a point in \mathcal{M} ($g \in G$).
- (b) This action is transitive, that is, for any two points y_1 and y_2 in \mathcal{M} , it is always possible to find a group element $g \in G$ such that $y_2 = gy_1$.

There is a one-to-one correspondence between the homogeneous spaces of G and the coset spaces of G . Let H_0 be a maximal subgroup of G which leaves the point y_0 invariant, $gy_0 = y_0$, $g \in H_0$, then H_0 is called the stabilizer of y_0 . Representing now any group element of G in the form $g = g_c g_0$, where $g_0 \in H_0$ and $g_c \in G/H_0$, we see that, by virtue of the transitivity property, any point $y \in \mathcal{M}$ can be given by $y = g_c g_0 y_0 = g_c y$. Hence, it follows that the elements g_c of the coset space give a parametrization of \mathcal{M} . The mapping $\mathcal{M} \leftrightarrow G/H_0$ is continuous since the group multiplication is continuous and the action on \mathcal{M} is continuous by definition. The stabilizers H and H_0 of two different points y and y_0 are conjugate, since from $H_0 g_0 = g_0$, $y_0 = g^{-1} y$, it follows that $g H_0 g^{-1} y = y$, that is, $H = g H_0 g^{-1}$.

Coming back to the Poincaré group \mathcal{P} , we see that the enumeration of the different homogeneous spaces \mathcal{M} of \mathcal{P} amounts to an enumeration of the subgroups of \mathcal{P} up to a conjugation. Following to Finkelstein, we require that \mathcal{M} always contains the Minkowski space $\mathbb{R}^{1,3}$ which means that four parameters of \mathcal{M} can be denoted by x (x^μ). This means that the stabilizer H of a given point in \mathcal{M} can never contain an element of the translation subgroup of \mathcal{P} . Thus, the stabilizer must be a subgroup of the proper Lorentz group \mathfrak{G}_+ .

In such a way, studying different subgroups of \mathfrak{G}_+ , we obtain a full list of homogeneous spaces $\mathcal{M} = \mathcal{P}/H$ of the Poincaré group. In the present paper, we restrict ourselves by a consideration of the following two homogeneous spaces:

$$\begin{aligned} \mathcal{M}_{10} &= \mathbb{R}^{1,3} \times \mathfrak{L}_6, & H &= 0; \\ \mathcal{M}_8 &= \mathbb{R}^{1,3} \times \mathbb{S}^2, & H &= \Omega_\psi^c; \end{aligned}$$

Hence, it follows that a group manifold of the Poincaré group, $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathfrak{L}_6$, is a maximal homogeneous space of \mathcal{P} , \mathfrak{L}_6 is a group manifold of the Lorentz group. The fields on the manifold \mathcal{M}_{10} were considered by Lurçat (1964). These fields depend on all the ten parameters of \mathcal{P} :

$$\psi(x, \mathfrak{g}) = \psi(x)\psi(\mathfrak{g}) = \psi(x_0, x_1, x_2, x_3)\psi(\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5, \mathfrak{g}_6),$$

where an explicit form of $\psi(x)$ is given by the exponentials, and the functions $\psi(\mathfrak{g})$ are expressed via the generalized hyperspherical functions $\mathfrak{M}_{mn;\bar{m}\bar{n}}^{ll}(\mathfrak{g})$ in the case of finite-dimensional representations.

The following eight-dimensional homogeneous space $\mathcal{M}_8 = \mathbb{R}^{1,3} \times \mathbb{S}^2$ is a direct product of the Minkowski space $\mathbb{R}^{1,3}$ and the complex two-sphere \mathbb{S}^2 . In this case, the stabilizer H consists of the subgroup Ω_ψ^c of the diagonal matrices $(\begin{smallmatrix} e^{\frac{i\psi^c}{2}} & 0 \\ 0 & e^{-\frac{i\psi^c}{2}} \end{smallmatrix})$. Bacry and Kihlberg (1969) claimed that the space \mathcal{M}_8 is the most suitable for a description of both half-integer and integer spins. The fields, defined in \mathcal{M}_8 , depend on the eight parameters of \mathcal{P} :

$$\psi(x, \varphi^c, \theta^c) = \psi(x)\psi(\varphi^c, \theta^c) = \psi(x_0, x_1, x_2, x_3)\psi(\varphi, \epsilon, \theta, \tau), \tag{46}$$

where the functions $\psi(\varphi^c, \theta^c)$ are expressed via the associated hyperspherical functions defined on the surface of the complex two-sphere \mathbb{S}^2 .

3.1. Harmonic Analysis on $SU(2) \otimes SU(2) \odot T_4$

In this subsection, we will consider Fourier series on the Poincaré group \mathcal{P} . First of all, the group \mathcal{P} has the same number of connected components as with the Lorentz group. Later on, we will consider only the component \mathcal{P}_+^\uparrow corresponding the connected component L_+^\uparrow (so called special Lorentz group, see (Rumer and Fet, 1977)). As is known, an universal covering $\overline{\mathcal{P}_+^\uparrow}$ of the group \mathcal{P}_+^\uparrow is defined by a semidirect product $\overline{\mathcal{P}_+^\uparrow} = SL(2, \mathbb{C}) \odot T_4 \sim\text{spin}_+(1, 3) \odot T_4$, where T_4 is a subgroup of four-dimensional translations. Since the Poincaré group is a 10-parameter group, then an invariant measure on this group has a form

$$d^{10}\alpha = d^6\mathfrak{g}d^4x,$$

where $d^6\mathfrak{g}$ is the Haar measure on the Lorentz group. Or, taking into account (51), we obtain

$$d\alpha = \sin\theta^c \sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon dx_1 dx_2 dx_3 dx_4, \tag{47}$$

where $x_i \in T_4$.

Thus, an invariant integration on the group $SL(2, \mathbb{C}) \odot T_4$ is defined by the formula

$$\int_{SL(2,\mathbb{C})\odot T_4} f(\alpha) d^{10}\alpha = \int_{SL(2,\mathbb{C})} \int_{T_4} f(x, \mathfrak{g}) d^4x d^6\mathfrak{g},$$

where $f(\alpha)$ is a finite continuous function on $SL(2, \mathbb{C}) \odot T_4$.

In the case of finite-dimensional representations we come again to a local isomorphism $SL(2, \mathbb{C}) \odot T_4 \simeq SU(2) \otimes SU(2) \odot T_4$. In this case, basis representation functions of the Poincaré group are defined by symmetric polynomials of the form

$$p(x, z, \bar{z}) = \sum_{\substack{(\alpha_1, \dots, \alpha_k) \\ (\dot{\alpha}_1, \dots, \dot{\alpha}_r)}} \frac{1}{k! r!} a^{\alpha_1 \dots \alpha_k \dot{\alpha}_1 \dots \dot{\alpha}_r}(x) z_{\alpha_1} \dots z_{\alpha_k} \bar{z}_{\dot{\alpha}_1} \dots \bar{z}_{\dot{\alpha}_r}, \tag{48}$$

$(\alpha_i, \dot{\alpha}_i = 0, 1)$

where the coefficients $a^{\alpha_1 \dots \alpha_k \dot{\alpha}_1 \dots \dot{\alpha}_r}$ depend on the variables x^α ($\alpha = 0, 1, 2, 3$; the parameters of T_4). The functions (48) should be considered as *the functions on the Poincaré group*. Some applications of these functions are contained in (Vasiliev, 1996; Gitman and Shelepin, 2001). The group T_4 is an Abelian group formed by a direct product of the four one-dimensional translation groups, T_1 , where T_1 is isomorphic to an additive group of real numbers \mathbb{R} (usual Fourier analysis is formulated in terms of the group \mathbb{R}). Hence it follows that all irreducible representations of T_4 are one-dimensional and expressed via the exponentials. Thus, the basis functions (matrix elements) of the finite-dimensional representations of \mathcal{P} have the form

$$t_{mn;\dot{m}\dot{n}}^{ll}(\alpha) = e^{-ipx} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll}(\mathfrak{g}), \tag{49}$$

where $x = (x_1, x_2, x_3, x_4)$, and $\mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll}(\mathfrak{g})$ is the generalized hyperspherical function (23).

Let us consider now the configuration space $\mathcal{M}_8 = \mathbb{R}^{1,3} \times \mathbb{S}^2$. In this case, the Fourier series on \mathcal{M}_8 can be defined as follows

$$f(\alpha) = \sum_{p=-\infty}^{+\infty} e^{ipx} \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{\dot{m}=-l}^{\dot{l}} \alpha_{m\dot{m}}^{ll} \mathfrak{M}_{l\dot{l}}^{m\dot{m}}(\varphi, \epsilon, \theta, \tau, 0, 0), \tag{50}$$

where

$$\alpha_{m\dot{m}}^{l\dot{l}} = \frac{(-1)^m(2l+1)(2\dot{l}+1)}{32\pi^4} \int_{\mathbb{S}^2} \int_{T_4} f(\alpha) e^{-ipx} \mathfrak{M}_{l\dot{l}}^{m\dot{m}}(\varphi, \epsilon, \theta, \tau, 0, 0) d^4x d^4\mathfrak{g},$$

and $d^4\mathfrak{g} = \sin\theta^c \sin\dot{\theta}^c d\theta d\varphi d\tau d\epsilon$ is the Haar measure on \mathbb{S}^2 , $f(\alpha)$ is the square integrable function on \mathcal{M}_8 , such that

$$\int_{\mathbb{S}^2} \int_{T_4} |f(\alpha)|^2 d^4x d^4\mathfrak{g} < +\infty.$$

3.2. Harmonic Analysis on the Group $SL(2, \mathbb{C})$

First of all, on the group $SL(2, \mathbb{C})$, there exists an invariant measure $d\mathfrak{g}$, that is, such a measure that for any finite continuous function $f(\mathfrak{g})$ on $SL(2, \mathbb{C})$ the following equality

$$\int f(\mathfrak{g}) d\mathfrak{g} = \int f(\mathfrak{g}_0\mathfrak{g}) d\mathfrak{g} = \int f(\mathfrak{g}\mathfrak{g}_0) d\mathfrak{g} = \int f(\mathfrak{g}^{-1}) d\mathfrak{g}$$

holds. Now we express the Haar measure (left or right) in terms of the parameters (1),

$$d\mathfrak{g} = \sin\theta^c \sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon. \tag{51}$$

Thus, an invariant integration on the group $SL(2, \mathbb{C})$ is defined by the formula

$$\int_{SL(2,\mathbb{C})} f(\mathfrak{g}) d\mathfrak{g} = \frac{1}{32\pi^4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi f(\theta, \varphi, \psi, \tau, \epsilon, \varepsilon) \times \sin\theta^c \sin\dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon.$$

When we consider finite-dimensional (spinor) representations of $SL(2, \mathbb{C})$, we come naturally to a local isomorphism $SU(2) \otimes SU(2) \simeq SL(2, \mathbb{C})$ considered by many authors (Huszar and Smorodinsky, 1970; Ryder, 1985). Since a dimension of the spinor representation $\tau_{l\dot{l}}$ of $SU(2) \otimes SU(2)$ is equal to $(2l+1)(2\dot{l}+1)$, then the functions $\sqrt{(2l+1)(2\dot{l}+1)} t_{m\dot{m};\dot{m}\dot{n}}^{l\dot{l}}(\mathfrak{g})$ form a full orthogonal normalized system on this group with respect to the invariant measure $d\mathfrak{g}$. At this point, the indices l and \dot{l} run all possible integer or half-integer nonnegative values, and the indices m, n and \dot{m}, \dot{n} run the values $-l, -l+1, \dots, l-1, l$ and $-\dot{l}, -\dot{l}+1, \dots, \dot{l}-1, \dot{l}$. In virtue of (23) the matrix elements $t_{m\dot{m};\dot{m}\dot{n}}^{l\dot{l}}$ are expressed via the generalized

hyperspherical function $t_{mn;\dot{m}\dot{n}}^{ll}(\mathbf{g}) = \mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll}(\varphi, \epsilon, \theta, \tau, \psi, \epsilon)$. Therefore,

$$\int_{SU(2) \otimes SU(2)} \mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll}(\mathbf{g}) \overline{\mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll}(\mathbf{g})} d\mathbf{g} = \frac{32\pi^4}{(2l+1)(2\dot{l}+1)} \delta(\mathbf{g}' - \mathbf{g}), \quad (52)$$

where $\delta(\mathbf{g}' - \mathbf{g})$ is a δ -function on the group $SU(2) \otimes SU(2)$. An explicit form of δ -function is

$$\begin{aligned} \delta(\mathbf{g}' - \mathbf{g}) &= \delta(\varphi' - \varphi) \delta(\epsilon' - \epsilon) \delta(\cos \theta' \cos h\tau' - \cos \theta \cos h\tau) \\ &\quad \times \delta(\sin \theta' \sin h\tau' - \sin \theta \sin h\tau) \delta(\psi' - \psi) \delta(\epsilon' - \epsilon) \end{aligned}$$

Substituting into (52), the expression

$$\mathfrak{M}_{mn;\dot{m}\dot{n}}^{ll}(\mathbf{g}) = e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}(\theta, \tau) e^{-\dot{m}(\epsilon-i\varphi)-\dot{n}(\epsilon-i\psi)}$$

and taking into account (51), we obtain

$$\begin{aligned} &\int_{SU(2) \otimes SU(2)} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}(\theta, \tau) \overline{\mathfrak{Z}_{pq;\dot{p}\dot{q}}^{s\dot{s}}(\theta, \tau)} e^{-(m+p)\epsilon} e^{-i(m-p)\varphi} \\ &\quad \times e^{-(\dot{m}+\dot{p})\epsilon} e^{i(\dot{m}-\dot{p})\varphi} \times e^{-(n+q)\epsilon} \times e^{-i(n-q)\psi} e^{-(\dot{n}+\dot{q})\epsilon} \\ &\quad \times e^{i(\dot{n}-\dot{q})\psi} \sin \theta^c \sin \dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\epsilon = \frac{32\pi^4 \delta_{ls} \delta_{\dot{l}\dot{s}} \delta_{mp} \delta_{nq} \delta_{\dot{m}\dot{p}} \delta_{\dot{n}\dot{q}} \delta(\mathbf{g}' - \mathbf{g})}{(2l+1)(2\dot{l}+1)}. \end{aligned}$$

Thus, any square integrable function $f(\varphi^c, \theta^c, \psi^c)$ on the group $SU(2) \otimes SU(2)$, such that

$$\int_{SU(2) \otimes SU(2)} |f(\varphi^c, \theta^c, \psi^c)|^2 \sin \theta^c \sin \dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\epsilon < +\infty,$$

is expanded into a convergent (on an average) Fourier series on $SU(2) \otimes SU(2)$,

$$\begin{aligned} f(\varphi^c, \theta^c, \psi^c) &= \sum_{l,\dot{l}=0}^{\infty} \sum_{m=-l}^l \sum_{n=-l}^l \sum_{\dot{m}=-\dot{l}}^{\dot{l}} \sum_{\dot{n}=-\dot{l}}^{\dot{l}} \alpha_{mn;\dot{m}\dot{n}}^{ll} \\ &\quad \times e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll}(\cos \theta^c, \cos \dot{\theta}^c) e^{-\dot{m}(\epsilon-i\varphi)-\dot{n}(\epsilon-i\psi)}, \quad (53) \end{aligned}$$

where

$$\begin{aligned} \alpha_{mn;\dot{m}\dot{n}}^{ll} &= \frac{(-1)^{m-n} (2l+1)(2\dot{l}+1)}{32\pi^4} \\ &\quad \times \int_{SU(2) \otimes SU(2)} f(\varphi^c, \theta^c, \psi^c) e^{i(m\varphi^c+n\psi^c)} \mathfrak{Z}_{mn;\dot{m}\dot{n}}^{ll} \\ &\quad \times (\cos \theta^c, \cos \dot{\theta}^c) e^{-i(\dot{m}\dot{\varphi}^c+\dot{n}\dot{\psi}^c)} \sin \theta^c \sin \dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\epsilon. \end{aligned}$$

The Parseval equality for the case of $SU(2) \otimes SU(2)$ is defined as follows

$$\sum_{l,l=0}^{\infty} \sum_{m=-l}^l \sum_{n=-l}^l \sum_{\dot{m}=-l}^l \sum_{\dot{n}=-l}^l |\alpha_{mn;\dot{m}\dot{n}}^{ll}|^2 = \frac{(2l+1)(2\dot{l}+1)}{32\pi^4} \int_{SU(2) \otimes SU(2)} |f(\varphi^c, \theta^c, \psi^c)|^2 \sin \theta^c \sin \dot{\theta}^c d\theta d\varphi d\psi d\tau d\epsilon d\varepsilon.$$

About convergence of Fourier series of the type (53) see Beers and Dragt (1970).

In a like manner, we can define Fourier series on the two-dimensional complex sphere via the associated hyperspherical functions. An expansion of the functions on the surface of the two-dimensional sphere has an important meaning for the subsequent physical applications.

So, let $f(\varphi^c, \theta^c)$ be a function on the complex two-sphere \mathbb{S}^2 , such that

$$\int_{\mathbb{S}^2} |f(\varphi^c, \theta^c)|^2 \sin \theta^c \sin \dot{\theta}^c d\theta d\varphi d\tau d\epsilon < +\infty,$$

then $f(\varphi^c, \theta^c)$ is expanded into a convergent Fourier series on \mathbb{S}^2 ,

$$f(\varphi^c, \theta^c) = \sum_{l,l=0}^{\infty} \sum_{m=-l}^l \sum_{\dot{m}=-l}^l \alpha_{m\dot{m}}^{ll} e^{-m(\epsilon+i\varphi)} \mathfrak{Z}_{l\dot{l}}^{m\dot{m}}(\cos \theta^c, \cos \dot{\theta}^c) e^{-\dot{m}(\epsilon-i\varphi)},$$

where

$$\alpha_{m\dot{m}}^{ll} = \frac{(-1)^m(2l+1)(2\dot{l}+1)}{32\pi^4} \int_{\mathbb{S}^2} f(\varphi^c, \theta^c) e^{im\varphi^c} \mathfrak{Z}_{l\dot{l}}^{m\dot{m}} \times (\cos \theta^c, \cos \dot{\theta}^c) e^{-i\dot{m}\varphi^c} \sin \theta^c \sin \dot{\theta}^c d\theta d\varphi d\tau d\epsilon,$$

and $\mathfrak{Z}_{l\dot{l}}^{m\dot{m}}(\cos \theta^c, \cos \dot{\theta}^c)$ is an associated hyperspherical function, $d\mathfrak{g} = \sin \theta^c \sin \dot{\theta}^c d\theta d\varphi d\tau d\epsilon$ is the Haar measure on the sphere \mathbb{S}^2 . Correspondingly, the Parseval equality on \mathbb{S}^2 has the form

$$\sum_{l,l=0}^{\infty} \sum_{m=-l}^l \sum_{\dot{m}=-l}^l |\alpha_{m\dot{m}}^{ll}|^2 = \frac{(2l+1)(2\dot{l}+1)}{32\pi^4} \int_{\mathbb{S}^2} |f(\varphi^c, \theta^c)|^2 \sin \theta^c \sin \dot{\theta}^c d\theta d\varphi d\tau d\epsilon.$$

4. LAGRANGIAN FORMALISM AND FIELD EQUATIONS ON THE POINCARÉ GROUP

We will start with a more general homogeneous space of the group \mathcal{P} , $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathcal{L}_6$ (group manifold of the Poincaré group). Let $\mathcal{L}(\alpha)$ be a Lagrangian on the group manifold \mathcal{M}_{10} (in other words, $\mathcal{L}(\alpha)$ is a 10-dimensional point function), where α is the parameter set of this group. Then an integral for $\mathcal{L}(\alpha)$ on some 10-dimensional volume Ω of the group manifold we will call *an action on the Poincaré group*:

$$A = \int_{\Omega} d\alpha \mathcal{L}(\alpha),$$

where $d\alpha$ is a Haar measure on the group \mathcal{P} (see (47)).

Let $\psi(\alpha)$ be a function on the group manifold \mathcal{M}_{10} (now it is sufficient to assume that $\psi(\alpha)$ is a square integrable function on the Poincaré group) and let

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial \alpha} \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi}{\partial \alpha}} = 0 \tag{54}$$

be Euler–Lagrange equations on \mathcal{M}_{10} [more precisely speaking, (54) act on the tangent bundle $T\mathcal{M}_{10} = \cup_{\alpha \in \mathcal{M}_{10}} \cup T_{\alpha} \mathcal{M}_{10}$ of the manifold \mathcal{M}_{10} , see Arnold (1989)]. Let us introduce a Lagrangian $\mathcal{L}(\alpha)$ depending on the field function $\psi(\alpha)$ as follows

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left(\psi^*(\alpha) B_{\mu} \frac{\partial \psi(\alpha)}{\partial \alpha_{\mu}} - \frac{\partial \psi^*(\alpha)}{\partial \alpha_{\mu}} B_{\mu} \psi(\alpha) \right) - \kappa \psi^*(\alpha) B_{11} \psi(\alpha),$$

where B_{ν} ($\nu = 1, 2, \dots, 10$) are square matrices. The number of rows and columns in these matrices is equal to the number of components of $\psi(\alpha)$, κ is a nonnull real constant.

Further, if B_{11} is nonsingular, then we can introduce the matrices

$$\Gamma_{\mu} = B_{11}^{-1} B_{\mu}, \quad \mu = 1, 2, \dots, 10,$$

and represent the Lagrangian $\mathcal{L}(\alpha)$ in the form

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left(\bar{\psi}(\alpha) \Gamma_{\mu} \frac{\partial \psi(\alpha)}{\partial \alpha_{\mu}} - \frac{\bar{\psi}(\alpha)}{\partial \alpha_{\mu}} \Gamma_{\mu} \psi(\alpha) \right) - \kappa \bar{\psi}(\alpha) \psi(\alpha), \tag{55}$$

where

$$\bar{\psi}(\alpha) = \psi^*(\alpha) B_{11}.$$

Varying independently $\psi(x)$ and $\bar{\psi}(x)$, we obtain from (55) in accordance with (54) the following equations:

$$\begin{aligned} \Gamma_i \frac{\partial \psi(x)}{\partial x_i} + \kappa \psi(x) &= 0, \quad (i = 1, \dots, 4) \\ \Gamma_i^T \frac{\partial \bar{\psi}(x)}{\partial x_i} - \kappa \bar{\psi}(x) &= 0. \end{aligned} \tag{56}$$

Analogously, varying independently $\psi(\mathfrak{g})$ and $\bar{\psi}(\mathfrak{g})$ one gets

$$\begin{aligned} \Gamma_k \frac{\partial \psi(\mathfrak{g})}{\partial \mathfrak{g}_k} + \kappa \psi(\mathfrak{g}) &= 0, \quad (k = 1, \dots, 6) \\ \Gamma_k^T \frac{\partial \bar{\psi}(\mathfrak{g})}{\partial \mathfrak{g}_k} - \kappa \bar{\psi}(\mathfrak{g}) &= 0, \end{aligned} \tag{57}$$

where

$$\psi(\mathfrak{g}) = \begin{pmatrix} \psi(\mathfrak{g}) \\ \dot{\psi}(\mathfrak{g}) \end{pmatrix}, \quad \Gamma_k = \begin{pmatrix} 0 & \Lambda_k^{*ll} \\ \Lambda_k^{ll} & 0 \end{pmatrix}.$$

The doubling of representations, described by a bispinor $\psi(\mathfrak{g}) = (\psi(\mathfrak{g}), \dot{\psi}(\mathfrak{g}))^T$, is the well known feature of the Lorentz group representations (Gel'fand *et al.*, 1963; Naimark, 1964). Since an universal covering $SL(2, \mathbb{C})$ of the proper orthochronous Lorentz group is a complexification of the group $SU(2)$ (see Section 2), then it is more convenient to express six parameters \mathfrak{g}_k of the Lorentz group via three parameters a_1, a_2, a_3 of the group $SU(2)$. It is obvious that $\mathfrak{g}_1 = a_1, \mathfrak{g}_2 = a_2, \mathfrak{g}_3 = a_3, \mathfrak{g}_4 = ia_1, \mathfrak{g}_5 = ia_2, \mathfrak{g}_6 = ia_3$. Then, the first equation from (57) can be written as

$$\begin{aligned} \sum_{j=1}^3 \Lambda_j^{ll} \frac{\partial \psi}{\partial a_j} - i \sum_{j=1}^3 \Lambda_j^{ll} \frac{\partial \psi}{\partial a_j^*} + k^c \dot{\psi} &= 0, \\ \sum_{j=1}^3 \Lambda_j^{*ll} \frac{\partial \dot{\psi}}{\partial \tilde{a}_j} + i \sum_{j=1}^3 \Lambda_j^{*ll} \frac{\partial \dot{\psi}}{\partial \tilde{a}_j^*} + k^c \psi &= 0. \end{aligned} \tag{58}$$

where $a_1^* = -i\mathfrak{g}_4, a_2^* = -i\mathfrak{g}_5, a_3^* = -i\mathfrak{g}_6$, and $\tilde{a}_j, \tilde{a}_j^*$ are the parameters corresponding the dual basis. In essence, the equations (58) are defined in a three-dimensional complex space \mathbb{C}^3 . In turn, the space \mathbb{C}^3 is isometric to a six-dimensional bivector space \mathbb{R}^6 (a parameter space of the Lorentz group (Kagan, 1926; Petrov, 1969). The bivector space \mathbb{R}^6 is a tangent space of the group manifold \mathfrak{L}_6 of the Lorentz group, that is, the manifold \mathfrak{L}_6 in the each its point is equivalent locally to the space \mathbb{R}^6 . Thus, for all $\mathfrak{g} \in \mathfrak{L}_6$ we have $T_{\mathfrak{g}}\mathfrak{L}_6 \simeq \mathbb{R}^6$. There exists a close relationship between the metrics of the Minkowski spacetime $\mathbb{R}^{1,3}$ and the metrics of \mathbb{R}^6 defined by the formulae (Petrov, 1969)

$$g_{ab} \longrightarrow g_{\alpha\beta\gamma\delta} \equiv g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}, \tag{59}$$

where $g_{\alpha\beta}$ is a metric tensor of the spacetime $\mathbb{R}^{1,3}$, and collective indices are skewsymmetric pairs $\alpha\beta \rightarrow a, \gamma\delta \rightarrow b$. In more detail, if

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then in virtue of (59) for the metric tensor of \mathbb{R}^6 , we obtain

$$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{60}$$

where the order of collective indices in \mathbb{R}^6 is $23 \rightarrow 0, 10 \rightarrow 1, 20 \rightarrow 2, 30 \rightarrow 3, 31 \rightarrow 4, 12 \rightarrow 5$.

Let us write an invariance condition for the system (58). As it is shown in Kagan (1926), the Lorentz transformations can be represented by linear transformations of the space \mathbb{R}^6 . Let $\mathfrak{g} : a' = \mathfrak{g}^{-1}a$ be a transformation of the bivector space \mathbb{R}^6 , that is, $a' = \sum_{b=1}^6 g_{ba}a_b$, where $a = (a_1, a_2, a_3, a_1^*, a_2^*, a_3^*)$ and g_{ba} is the metric tensor (60). We can write the tensor (60) in the form $g_{ab} = \begin{pmatrix} g_{ik}^- \\ g_{ik}^+ \end{pmatrix}$, then $a' = \sum_{k=1}^3 g_{ki}^- a_k, a'^* = \sum_{k=1}^3 g_{ki}^+ a_k^*$. Replacing ψ via $T_{li}^{-1}(\mathfrak{g})\psi'$, and differentiation on $a_k (a_k^*)$ by differentiation on $a'_k (a_k'^*)$ via the formulae

$$\frac{\partial}{\partial a_k} = \sum g_{ik}^- \frac{\partial}{\partial a'_i}, \quad \frac{\partial}{\partial a_k^*} = \sum g_{ik}^+ \frac{\partial}{\partial a_k'^*},$$

we obtain

$$\begin{aligned} & \sum_{i=1}^3 \left[g_{i1}^- \Lambda_1^{ii} \frac{\partial(T_{li}^{-1}(\mathfrak{g})\psi')}{\partial a'_i} + g_{i2}^- \Lambda_2^{ii} \frac{\partial(T_{li}^{-1}(\mathfrak{g})\psi')}{\partial a'_i} + g_{i3}^- \Lambda_3^{ii} \frac{\partial(T_{li}^{-1}(\mathfrak{g})\psi')}{\partial a'_i} \right. \\ & \left. - i g_{i1}^- \Lambda_1^{ii} \frac{\partial(T_{li}^{-1}(\mathfrak{g})\psi')}{\partial a_i'^*} - i g_{i2}^- \Lambda_2^{ii} \frac{\partial(T_{li}^{-1}(\mathfrak{g})\psi')}{\partial a_i'^*} - i g_{i3}^- \Lambda_3^{ii} \frac{\partial(T_{li}^{-1}(\mathfrak{g})\psi')}{\partial a_i'^*} \right] \\ & + \kappa^c T_{li}^{-1}(\mathfrak{g})\psi' = 0, \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^3 \left[g_{i1}^+ \Lambda_1^{*li} \frac{\partial (T_{ii}^{*-1}(\mathfrak{g})\psi')}{\partial \tilde{a}'_i} + g_{i2}^+ \Lambda_2^{*li} \frac{\partial (T_{ii}^{*-1}(\mathfrak{g})\psi')}{\partial \tilde{a}'_i} + g_{i3}^+ \Lambda_3^{*li} \frac{\partial (T_{ii}^{*-1}(\mathfrak{g})\psi')}{\partial \tilde{a}'_i} \right. \\ & \left. + i g_{i1}^+ \Lambda_1^{*li} \frac{\partial (T_{ii}^{*-1}(\mathfrak{g})\psi')}{\partial \tilde{a}^{*i'}} + i g_{i2}^+ \Lambda_2^{*li} \frac{\partial (T_{ii}^{*-1}(\mathfrak{g})\psi')}{\partial \tilde{a}^{*i'}} + i g_{i3}^+ \Lambda_3^{*li} \frac{\partial (T_{ii}^{*-1}(\mathfrak{g})\psi')}{\partial \tilde{a}^{*i'}} \right] \\ & + \kappa^c T_{ii}^{*-1}(\mathfrak{g})\psi' = 0. \end{aligned}$$

or,

$$\begin{aligned} & \sum_i \left[g_{i1}^- \Lambda_1^{li} T_{ii}^{-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial a'_i} + g_{i2}^- \Lambda_2^{li} T_{ii}^{-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial a'_i} + g_{i3}^- \Lambda_3^{li} T_{ii}^{-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial a'_i} \right. \\ & \left. - i g_{i1}^- \Lambda_1^{li} T_{ii}^{-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial a_i^{*'}} - i g_{i2}^- \Lambda_2^{li} T_{ii}^{-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial a_i^{*'}} - i g_{i3}^- \Lambda_3^{li} T_{ii}^{-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial a_i^{*'}} \right] \\ & + \kappa^c T_{ii}^{-1}(\mathfrak{g})\psi' = 0, \\ & \sum_i \left[g_{i1}^+ \Lambda_1^{*li} T_{ii}^{*-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial \tilde{a}'_i} + g_{i2}^+ \Lambda_2^{*li} T_{ii}^{*-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial \tilde{a}'_i} + g_{i3}^+ \Lambda_3^{*li} T_{ii}^{*-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial \tilde{a}'_i} \right. \\ & \left. + i g_{i1}^+ \Lambda_1^{*li} T_{ii}^{*-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial \tilde{a}^{*i'}} + i g_{i2}^+ \Lambda_2^{*li} T_{ii}^{*-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial \tilde{a}^{*i'}} + i g_{i3}^+ \Lambda_3^{*li} T_{ii}^{*-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial \tilde{a}^{*i'}} \right] \\ & + \kappa^c T_{ii}^{*-1}(\mathfrak{g})\psi' = 0. \end{aligned}$$

For coincidence of the latter system with (58), we must multiply this system by $T_{ii}(\mathfrak{g}) (T_{ii}^*(\mathfrak{g}))$ from the left,

$$\begin{aligned} & \sum_i \sum_k g_{ik}^- T_{ii}(\mathfrak{g}) \Lambda_k^{li} T_{ii}^{-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial a'_i} \\ & - i \sum_i \sum_k g_{ik}^- T_{ii}(\mathfrak{g}) \Lambda_k^{li} T_{ii}^{-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial a_i^{*'}} + \kappa^c \psi' = 0, \\ & \sum_i \sum_k g_{ik}^+ T_{ii}^*(\mathfrak{g}) \Lambda_k^{*li} T_{ii}^{*-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial \tilde{a}'_i} \\ & + i \sum_i \sum_k g_{ik}^+ T_{ii}^*(\mathfrak{g}) \Lambda_k^{*li} T_{ii}^{*-1}(\mathfrak{g}) \frac{\partial \psi'}{\partial \tilde{a}^{*i'}} + \kappa^c \psi' = 0. \end{aligned}$$

The requirement of invariance means that for any transformation \mathfrak{g} between the matrices Λ_k^{ij} (Λ_k^{*ij}) the following relations hold:

$$\begin{aligned} \sum_k g_{ik}^- T_{li}(\mathfrak{g}) \Lambda_k^{ij} T_{li}^{-1}(\mathfrak{g}) &= \Lambda_i^{ij}, \\ \sum_k g_{ik}^+ T_{li}^*(\mathfrak{g}) \Lambda_k^{*ij} T_{li}^{*-1}(\mathfrak{g}) &= \Lambda_i^{*ij}, \end{aligned} \tag{61}$$

where Λ_i^{*ij} are the matrices of the equations in the dual representation space, κ^c is a complex number, $\partial/\partial\tilde{a}_i$ mean covariant derivatives in the dual space.

5. THE STRUCTURE OF THE MATRICES Λ_i^{ij}

First of all, let us find commutation relations between the matrices Λ_i^{ij} , Λ_i^{*ij} and infinitesimal operators (8)–(19) defined in the helicity basis. Let us represent transformations $T_{li}(\mathfrak{g})$ ($T_{li}^*(\mathfrak{g})$) in the infinitesimal form,

$$\begin{aligned} 1 + A_i^{ij} \xi + \dots, & \quad 1 + B_i^{ij} \xi + \dots, \\ 1 + \tilde{A}_i^{ij} \xi + \dots, & \quad 1 + \tilde{B}_i^{ij} \xi + \dots \end{aligned}$$

It is easy to see that the bivector space \mathbb{R}^6 contains two three-dimensional subspaces \mathbb{R}_-^3 and \mathbb{R}_+^3 with the metric tensors g_{ik}^- and g_{ik}^+ , respectively. Let us consider a rotation $\mathfrak{g} = e + a_1 \xi + \dots$ in the subspace \mathbb{R}_-^3 . The matrix of this rotation can be represented in the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\xi \\ 0 & \xi & 1 \end{pmatrix}.$$

Substituting these transformations into invariance conditions (61), we obtain with an accuracy of the terms of second order the following three equalities:

$$\begin{aligned} (1 + A_1^{ij} \xi) \Lambda_1^{ij} (1 - A_1^{ij} \xi) &= \Lambda_1^{ij}, \\ (1 + A_1^{ij} \xi) (\Lambda_2^{ij} - \xi \Lambda_3^{ij}) (1 - A_1^{ij} \xi) &= \Lambda_2^{ij}, \\ (1 + A_1^{ij} \xi) (\xi \Lambda_2^{ij} + \Lambda_3^{ij}) (1 - A_1^{ij} \xi) &= \Lambda_3^{ij}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} A_1^{ii} \Lambda_1^{ii} - \Lambda_1^{ii} A_1^{ii} &= 0, \\ A_1^{ii} \Lambda_2^{ii} - \Lambda_2^{ii} A_1^{ii} - \Lambda_3^{ii} &= 0, \\ A_1^{ii} \Lambda_3^{ii} - \Lambda_3^{ii} A_1^{ii} + \Lambda_2^{ii} &= 0, \end{aligned}$$

or

$$\begin{aligned} [A_1^{ii}, \Lambda_1^{ii}] &= 0, \\ [A_1^{ii}, \Lambda_2^{ii}] &= \Lambda_3^{ii}, \\ [A_1^{ii}, \Lambda_3^{ii}] &= -\Lambda_2^{ii}. \end{aligned}$$

Analogously, for a rotation $\mathfrak{g} = e + a_2 \xi + \dots$ with the matrix

$$\begin{pmatrix} 1 & 0 & \xi \\ 0 & 1 & 0 \\ -\xi & 0 & 1 \end{pmatrix}$$

we have

$$\begin{aligned} (1 + A_2^{ii} \xi)(\Lambda_1^{ii} + \xi \Lambda_3^{ii})(1 - A_2^{ii} \xi) &= \Lambda_1^{ii}, \\ (1 + A_2^{ii} \xi) \Lambda_2^{ii} (1 - A_2^{ii} \xi) &= \Lambda_2^{ii}, \\ (1 + A_2^{ii} \xi)(\xi \Lambda_1^{ii} + \Lambda_3^{ii})(1 - A_2^{ii} \xi) &= \Lambda_3^{ii}. \end{aligned}$$

From the latter relations, we see that

$$\begin{aligned} [A_2^{ii}, \Lambda_1^{ii}] &= -\Lambda_3^{ii}, \\ [A_2^{ii}, \Lambda_2^{ii}] &= 0, \\ [A_2^{ii}, \Lambda_3^{ii}] &= \Lambda_1^{ii}. \end{aligned}$$

Further, taking into account all possible transformations (rotations) in the subspaces \mathbb{R}_-^3 and \mathbb{R}_+^3 , we obtain the following commutation relations:

$$\begin{aligned} [A_1^{ii}, \Lambda_1^{ii}] &= 0, & [A_1^{ii}, \Lambda_2^{ii}] &= \Lambda_3^{ii}, & [A_1^{ii}, \Lambda_3^{ii}] &= -\Lambda_2^{ii}, \\ [A_2^{ii}, \Lambda_1^{ii}] &= -\Lambda_3^{ii}, & [A_2^{ii}, \Lambda_2^{ii}] &= 0, & [A_2^{ii}, \Lambda_3^{ii}] &= \Lambda_1^{ii}, \\ [A_3^{ii}, \Lambda_1^{ii}] &= \Lambda_2^{ii}, & [A_3^{ii}, \Lambda_2^{ii}] &= -\Lambda_1^{ii}, & [A_3^{ii}, \Lambda_3^{ii}] &= 0. \end{aligned} \quad (62)$$

$$\begin{aligned}
 [B_1^{ii}, \Lambda_1^{ii}] &= 0, & [B_1^{ii}, \Lambda_2^{ii}] &= -i\Lambda_3^{ii}, & [B_1^{ii}, \Lambda_3^{ii}] &= i\Lambda_2^{ii}, \\
 [B_2^{ii}, \Lambda_1^{ii}] &= i\Lambda_3^{ii}, & [B_2^{ii}, \Lambda_2^{ii}] &= 0, & [B_2^{ii}, \Lambda_3^{ii}] &= -i\Lambda_1^{ii}, \\
 [B_3^{ii}, \Lambda_1^{ii}] &= -i\Lambda_2^{ii}, & [B_3^{ii}, \Lambda_2^{ii}] &= i\Lambda_1^{ii}, & [B_3^{ii}, \Lambda_3^{ii}] &= 0.
 \end{aligned} \tag{63}$$

$$\begin{aligned}
 [\tilde{A}_1^{ii}, \Lambda_1^{ii}] &= 0, & [\tilde{A}_1^{ii}, \Lambda_2^{ii}] &= \Lambda_3^{ii}, & [\tilde{A}_1^{ii}, \Lambda_3^{ii}] &= -\Lambda_2^{ii}, \\
 [\tilde{A}_2^{ii}, \Lambda_1^{ii}] &= -\Lambda_3^{ii}, & [\tilde{A}_2^{ii}, \Lambda_2^{ii}] &= 0, & [\tilde{A}_2^{ii}, \Lambda_3^{ii}] &= \Lambda_1^{ii}, \\
 [\tilde{A}_3^{ii}, \Lambda_1^{ii}] &= \Lambda_2^{ii}, & [\tilde{A}_3^{ii}, \Lambda_2^{ii}] &= -\Lambda_1^{ii}, & [\tilde{A}_3^{ii}, \Lambda_3^{ii}] &= 0.
 \end{aligned} \tag{64}$$

$$\begin{aligned}
 [\tilde{B}_1^{ii}, \Lambda_1^{ii}] &= 0, & [\tilde{B}_1^{ii}, \Lambda_2^{ii}] &= i\Lambda_3^{ii}, & [\tilde{B}_1^{ii}, \Lambda_3^{ii}] &= -i\Lambda_2^{ii}, \\
 [\tilde{B}_2^{ii}, \Lambda_1^{ii}] &= -i\Lambda_3^{ii}, & [\tilde{B}_2^{ii}, \Lambda_2^{ii}] &= 0, & [\tilde{B}_2^{ii}, \Lambda_3^{ii}] &= i\Lambda_1^{ii}, \\
 [\tilde{B}_3^{ii}, \Lambda_1^{ii}] &= i\Lambda_2^{ii}, & [\tilde{B}_3^{ii}, \Lambda_2^{ii}] &= -i\Lambda_1^{ii}, & [\tilde{B}_3^{ii}, \Lambda_3^{ii}] &= 0.
 \end{aligned} \tag{65}$$

$$\begin{aligned}
 [\tilde{A}_1^{ii}, \Lambda_1^{*ii}] &= 0, & [\tilde{A}_1^{ii}, \Lambda_2^{*ii}] &= \Lambda_3^{*ii}, & [\tilde{A}_1^{ii}, \Lambda_3^{*ii}] &= -\Lambda_2^{*ii}, \\
 [\tilde{A}_2^{ii}, \Lambda_1^{*ii}] &= -\Lambda_3^{*ii}, & [\tilde{A}_2^{ii}, \Lambda_2^{*ii}] &= 0, & [\tilde{A}_2^{ii}, \Lambda_3^{*ii}] &= \Lambda_1^{*ii}, \\
 [\tilde{A}_3^{ii}, \Lambda_1^{*ii}] &= \Lambda_2^{*ii}, & [\tilde{A}_3^{ii}, \Lambda_2^{*ii}] &= -\Lambda_1^{*ii}, & [\tilde{A}_3^{ii}, \Lambda_3^{*ii}] &= 0.
 \end{aligned} \tag{66}$$

$$\begin{aligned}
 [\tilde{B}_1^{ii}, \Lambda_1^{*ii}] &= 0, & [\tilde{B}_1^{ii}, \Lambda_2^{*ii}] &= i\Lambda_3^{*ii}, & [\tilde{B}_1^{ii}, \Lambda_3^{*ii}] &= -i\Lambda_2^{*ii}, \\
 [\tilde{B}_2^{ii}, \Lambda_1^{*ii}] &= -i\Lambda_3^{*ii}, & [\tilde{B}_2^{ii}, \Lambda_2^{*ii}] &= 0, & [\tilde{B}_2^{ii}, \Lambda_3^{*ii}] &= i\Lambda_1^{*ii}, \\
 [\tilde{B}_3^{ii}, \Lambda_1^{*ii}] &= i\Lambda_2^{*ii}, & [\tilde{B}_3^{ii}, \Lambda_2^{*ii}] &= -i\Lambda_1^{*ii}, & [\tilde{B}_3^{ii}, \Lambda_3^{*ii}] &= 0.
 \end{aligned} \tag{67}$$

$$\begin{aligned}
 [A_1^{ii}, \Lambda_1^{*ii}] &= 0, & [A_1^{ii}, \Lambda_2^{*ii}] &= \Lambda_3^{*ii}, & [A_1^{ii}, \Lambda_3^{*ii}] &= -\Lambda_2^{*ii}, \\
 [A_2^{ii}, \Lambda_1^{*ii}] &= -\Lambda_3^{*ii}, & [A_2^{ii}, \Lambda_2^{*ii}] &= 0, & [A_2^{ii}, \Lambda_3^{*ii}] &= \Lambda_1^{*ii}, \\
 [A_3^{ii}, \Lambda_1^{*ii}] &= \Lambda_2^{*ii}, & [A_3^{ii}, \Lambda_2^{*ii}] &= -\Lambda_1^{*ii}, & [A_3^{ii}, \Lambda_3^{*ii}] &= 0.
 \end{aligned} \tag{68}$$

$$\begin{aligned}
 [B_1^{ii}, \Lambda_1^{*ii}] &= 0, & [B_1^{ii}, \Lambda_2^{*ii}] &= -i\Lambda_3^{*ii}, & [B_1^{ii}, \Lambda_3^{*ii}] &= i\Lambda_2^{*ii}, \\
 [B_2^{ii}, \Lambda_1^{*ii}] &= i\Lambda_3^{*ii}, & [B_2^{ii}, \Lambda_2^{*ii}] &= 0, & [B_2^{ii}, \Lambda_3^{*ii}] &= -i\Lambda_1^{*ii}, \\
 [B_3^{ii}, \Lambda_1^{*ii}] &= -i\Lambda_2^{*ii}, & [B_3^{ii}, \Lambda_2^{*ii}] &= i\Lambda_1^{*ii}, & [B_3^{ii}, \Lambda_3^{*ii}] &= 0.
 \end{aligned} \tag{69}$$

From the latter relations and definition as in (3), it immediately follows that commutation relations between $\Lambda_3^{li}, \Lambda_3^{*li}$ and generators $Y_{\pm}^{li}, Y_3^{li}, X_{\pm}^{li}, X_3^{li}$ are of the form

$$\begin{aligned}
 [[\Lambda_3^{li}, X_{-}^{li}], X_{+}^{li}] &= 2\Lambda_3^{li}, & [[l_*^d \Lambda_3^{li}, Y_{-}^{li}], Y_{+}^{li}] &= {}_2^* \Lambda_3^{li}, \\
 [[\Lambda_3^{li}, Y_{-}^{li}], Y_{+}^{li}] &= 2\Lambda_3^{li}, & [[l_*^d \Lambda_3^{li}, X_{-}^{li}], X_{+}^{li}] &= {}_2^* \Lambda_3^{li}, \\
 [\Lambda_3^{li}, X_3^{li}] &= 0, & [l_*^d \Lambda_3^{li}, Y_3^{li}] &= 0 \\
 [\Lambda_3^{li}, Y_3^{li}] &= 0, & [l_*^d \Lambda_3^{li}, X_3^{li}] &= 0
 \end{aligned}
 \tag{70}$$

Using the relations (70), we will find an explicit form of the matrices Λ_3^{li} and Λ_3^{*li} , and further we will find $\Lambda_1^{li}, \Lambda_2^{li}$ and $\Lambda_1^{*li}, \Lambda_2^{*li}$.

The wave function ψ is transformed within some representation $T_{li}(\mathfrak{g})$ of the group \mathfrak{G}_+ . We assume that $T_{li}(\mathfrak{g})$ is decomposed into irreducible representations. We will numerate the components of the function ψ by the indices l, \dot{l} and m, \dot{m} , where l (\dot{l}) is a weight of irreducible representation, m (\dot{m}) is a number of the components in the representation of the weight l (\dot{l}). In the case when a representation with one and the same weight l (\dot{l}) at the decomposition of ψ occurs more than one time, then with the aim to distinguish these representations we will add the index k (\dot{k}), which indicates a number of the representations of the weight l (\dot{l}). Denoting $\zeta_{lm;\dot{l}\dot{m}} = |lm; \dot{l}\dot{m}\rangle$ and coming to the helicity basis, we obtain a following decomposition for the wave function:

$$\psi(a_1, a_2, a_3, a_1^*, a_2^*, a_3^*) = \sum_{l,m,k,\dot{l},\dot{m},\dot{k}} \psi_{lm;\dot{l}\dot{m}}^{k\dot{k}}(a_1, a_2, a_3, a_1^*, a_2^*, a_3^*) \zeta_{lm;\dot{l}\dot{m}}^{k\dot{k}},$$

where $a_1, a_2, a_3, a_1^*, a_2^*, a_3^*$ are the coordinates of the complex space $\mathbb{C}^3 \simeq \mathbb{R}^6$ (parameters of $SL(2, \mathbb{C})^2$). Analogously, for the dual representation, we have

$$\dot{\psi}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_1^*, \tilde{a}_2^*, \tilde{a}_3^*) = \sum_{l,m,k,\dot{l},\dot{m},\dot{k}} \dot{\psi}_{lm;\dot{l}\dot{m}}^{k\dot{k}}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_1^*, \tilde{a}_2^*, \tilde{a}_3^*) \zeta_{lm;\dot{l}\dot{m}}^{k\dot{k}}.$$

The transformation Λ_3^{li} in the helicity basis has the form

$$\Lambda_3^{li} \zeta_{lm;\dot{l}\dot{m}}^{k\dot{k}} = \sum_{l',m',k',\dot{l}',\dot{m}',\dot{k}'} c_{l'l',m'm';\dot{l}',\dot{m}'\dot{m}}^{k'k;\dot{k}'\dot{k}} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'\dot{k}'}.$$

² Recall that the wave function $\psi(a_j, a_j^*)$ is defined on the group manifold \mathcal{L}_6 , that is, ψ is a function on the Lorentz group.

Using the commutators (70), we will find the numbers $c_{l',m';l',\dot{l},\dot{m}'\dot{m}}^{k'k;k'k}$. First of all, recalling that

$$\begin{aligned} \chi_{-}^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \alpha_m^l \zeta_{l,m-1;\dot{l}\dot{m}}^{kk}, \\ \chi_{+}^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \alpha_{m+1}^l \zeta_{l,m+1;\dot{l}\dot{m}}^{kk}, \\ \chi_3^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= m \zeta_{lm;\dot{l}\dot{m}}^{kk}, \end{aligned}$$

where $(\alpha_m^l)^2 = (l+m)(l-m+1)$, we obtain

$$\begin{aligned} \Lambda_3^{l\dot{l}} \chi_3^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= m \Lambda_3^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} = m \sum_{l',m',k',\dot{l}',\dot{m}',k'} c_{l',m';\dot{l}',\dot{l},\dot{m}'\dot{m}}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'}, \\ \chi_3^{l\dot{l}} \Lambda_3^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \chi_3^{l\dot{l}} \sum_{l',m',k',\dot{l}',\dot{m}',k'} c_{l',m';\dot{l}',\dot{l},\dot{m}'\dot{m}}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'} \\ &= m' \sum_{l',m',k',\dot{l}',\dot{m}',k'} c_{l',m';\dot{l}',\dot{l},\dot{m}'\dot{m}}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'}. \end{aligned}$$

From the second equation of (70), we have $(\Lambda_3^{l\dot{l}} \chi_3^{l\dot{l}} - \chi_3^{l\dot{l}} \Lambda_3^{l\dot{l}}) \zeta_{lm;\dot{l}\dot{m}}^{kk} = 0$. Therefore,

$$\sum_{l',m',k',\dot{l}',\dot{m}',k'} (m - m') c_{l',m';\dot{l}',\dot{l},\dot{m}'\dot{m}}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'} = 0.$$

Hence, it immediately follows that $m' = m$. By this reason, we can denote the coefficients $c_{l',m';\dot{l}',\dot{l},\dot{m}'\dot{m}}^{k'k;k'k}$ via $c_{l',m';\dot{l}',\dot{l},\dot{m}'\dot{m}}^{k'k;k'k}$. Analogously, from the relations

$$\begin{aligned} \Upsilon_{-}^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \alpha_{\dot{m}}^{\dot{l}} \zeta_{lm;\dot{l},\dot{m}-1}^{kk}, \\ \Upsilon_{+}^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \alpha_{\dot{m}+1}^{\dot{l}} \zeta_{lm;\dot{l},\dot{m}+1}^{kk}, \\ \Upsilon_3^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \dot{m} \zeta_{lm;\dot{l}\dot{m}}^{kk} \end{aligned}$$

and equation $(\Lambda_3^{l\dot{l}} \Upsilon_3^{l\dot{l}} - \Upsilon_3^{l\dot{l}} \Lambda_3^{l\dot{l}}) \zeta_{lm;\dot{l}\dot{m}}^{kk} = 0$, we see that the coefficients $c_{l',m';\dot{l}',\dot{l},\dot{m}'\dot{m}}^{k'k;k'k}$ can be replaced by $c_{l',m';\dot{l}',\dot{l},\dot{m}}^{k'k;k'k}$.

Let us now use the first equation of the system (70):

$$\Lambda_3^{l\dot{l}} \chi_{-}^{l\dot{l}} \zeta_{lm;\dot{l}\dot{m}}^{kk} = \Lambda_3^{l\dot{l}} \alpha_m^l \zeta_{l,m-1;\dot{l}\dot{m}}^{kk} = \alpha_m^l \sum_{l',k',\dot{l}',k'} c_{l',m-1;\dot{l}'\dot{m}}^{k'k;k'k} \zeta_{l',m-1;\dot{l}'\dot{m}}^{k'k'}$$

$$\begin{aligned} X_-^{li} \Lambda_3^{li} \zeta_{lm;i\dot{m}}^{kk} &= X_-^{li} \sum_{l',k',\dot{l},\dot{k}'} c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}} \zeta_{l',m;i',\dot{m}}^{k'\dot{k}} = \sum_{l',k',\dot{l},\dot{k}'} \alpha_m^{l'} c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}} \zeta_{l',m-1;i',\dot{m}}^{k'\dot{k}}, \\ [\Lambda_3^{li}, X_-^{li}] \zeta_{lm;i\dot{m}}^{kk} &= \sum_{l',k',\dot{l},\dot{k}'} [\alpha_m^l c_{l',m-1;i',\dot{m}}^{k'k;k'\dot{k}} - \alpha_m^{l'} c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}}] \zeta_{l',m-1;i',\dot{m}}^{k'\dot{k}}. \end{aligned}$$

Further,

$$\begin{aligned} [\Lambda_3^{li}, X_-^{li}] X_+^{li} \zeta_{lm;i\dot{m}}^{kk} &= [\Lambda_3^{li}, X_-^{li}] \alpha_{m+1}^l \zeta_{l,m+1;i\dot{m}}^{kk} \\ &= \alpha_{m+1}^l \sum_{l',k',\dot{l},\dot{k}'} [\alpha_{m+1}^l c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}} - \alpha_{m+1}^{l'} c_{l',m+1;i',\dot{m}}^{k'k;k'\dot{k}}] \zeta_{l',m;i',\dot{m}}^{k'\dot{k}}, \\ X_+^{li} [\Lambda_3^{li}, X_-^{li}] \zeta_{lm;i\dot{m}}^{kk} &= X_+^{li} \sum_{l',k',\dot{l},\dot{k}'} [\alpha_m^l c_{l',m-1;i',\dot{m}}^{k'k;k'\dot{k}} - \alpha_m^{l'} c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}}] \zeta_{l',m-1;i',\dot{m}}^{k'\dot{k}} \\ &= \sum_{l',k',\dot{l},\dot{k}'} \alpha_m^{l'} [\alpha_m^l c_{l',m-1;i',\dot{m}}^{k'k;k'\dot{k}} - \alpha_m^{l'} c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}}] \zeta_{l',m;i',\dot{m}}^{k'\dot{k}}, \end{aligned}$$

$$\begin{aligned} [[\Lambda_3^{li}, X_-^{li}] X_+^{li}] \zeta_{lm;i\dot{m}}^{kk} &= \sum_{l',k',\dot{l},\dot{k}'} \left\{ [(\alpha_{m+1}^l)^2 + (\alpha_m^{l'})^2] c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}} \right. \\ &\quad \left. - \alpha_m^{l'} \alpha_m^l c_{l',m-1;i',\dot{m}}^{k'k;k'\dot{k}} - \alpha_{m+1}^{l'} \alpha_{m+1}^l c_{l',m+1;i',\dot{m}}^{k'k;k'\dot{k}} \right\} \zeta_{l',m;i',\dot{m}}^{k'\dot{k}}. \end{aligned}$$

Thus, the first commutator from (70) gives a system of equations with respect to the coefficients $c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}}$:

$$\begin{aligned} 2c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}} &= [(\alpha_{m+1}^l)^2 + (\alpha_m^{l'})^2] c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}} - \alpha_m^{l'} \alpha_m^l c_{l',m-1;i',\dot{m}}^{k'k;k'\dot{k}} \\ &\quad - \alpha_{m+1}^{l'} \alpha_{m+1}^l c_{l',m+1;i',\dot{m}}^{k'k;k'\dot{k}}. \end{aligned}$$

Or, substituting their values instead of α_m^l , we obtain

$$\begin{aligned} 2c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}} &= [(l+m+1)(l-m) + (l'+m)(l'-m+1)] c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}} \\ &\quad - \sqrt{(l'+m)(l'-m+1)(l+m)(l-m+1)} c_{l',m-1;i',\dot{m}}^{k'k;k'\dot{k}} \\ &\quad - \sqrt{(l'+m+1)(l'-m)(l+m+1)(l-m)} c_{l',m+1;i',\dot{m}}^{k'k;k'\dot{k}}. \end{aligned} \tag{71}$$

This system can be solved at the fixed indices $l', l, \dot{l}, \dot{l}', k', k, \dot{k}, \dot{k}'$. Let us fix some indices $l', l, \dot{l}, \dot{l}', k', k, \dot{k}, \dot{k}'$ and denote $c_{l',m;i',\dot{m}}^{k'k;k'\dot{k}}$ via c_m . Then, we

obtain a system of homogeneous equations for c_m , where $-\min(l', l) \leq m \leq \min(l', l)$. We solve these equations using the Gauss method. When m has the value $m_0 = \min(l', l)$, we obtain an equation containing two unknown variables c_{m_0} and c_{m_0-1} , from which c_{m_0-1} is defined via c_{m_0} . Further, when m has a value $m_0 - 1$, we obtain an equation with c_{m_0-2} , c_{m_0-1} , c_{m_0} , from which we can define c_{m_0-2} via c_{m_0} again. In doing so, we see that the coefficients $c_{l', m; i', \dot{m}}^{k'k; \dot{k}'\dot{k}}$ different from zero when $|l' - l| \leq 1$, that is, at $l' = l$, $l' = l - 1$ and $l' = l + 1$. For other values of l' , the coefficients $c_{l', m; i', \dot{m}}^{k'k; \dot{k}'\dot{k}}$ are equal to zero. First, we take $l' = l$ ($i', i, k', k, \dot{k}', \dot{k}$ are arbitrary), then the equations (71) are rewritten as follows

$$[2 - (l + m + 1)(l - m) - (l - m)(l - m + 1)] c_{l, m; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} + (l + m)(l - m + 1) c_{l, m-1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} + (l + m + 1)(l - m) c_{l, m+1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} = 0.$$

Supposing $m = l$, we find that $(1 - l) c_{l, l; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} + l c_{l, l-1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} = 0$. Whence $c_{l, l; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} = c_{l, l-1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} \cdot l$; $c_{l, l-1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} = c_{l, l; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} (l - 1)$, where the constant $c_{l, l; i', \dot{m}}^{k'k; \dot{k}'\dot{k}}$ does not depend on m . Supposing $m = l - 1$, we find analogously that $c_{l, l-2; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} = c_{l, l-1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} (l - 2)$. It is easy to verify that for any m there is an equality

$$c_{l, m; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} = c_{l, l; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} \cdot m.$$

Let us suppose now $l' = l - 1$, then the equations (71) take the form

$$[2 - (l + m + 1)(l - m) - (l + m - 1)(l - m)] c_{l-1, l, m; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} + \sqrt{(l + m - 1)(l - m)(l + m)(l - m + 1)} c_{l-1, l, m-1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} + \sqrt{(l + m)(l - m - 1)(l + m + 1)(l - m)} c_{l-1, l, m+1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} = 0.$$

Making in these equations the substitutions

$$\begin{aligned} c_{l-1, l, m; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} &= \tilde{c}_{l-1, l, m; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} \sqrt{(l + m)(l - m)}, \\ c_{l-1, l, m-1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} &= \tilde{c}_{l-1, l, m-1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} \sqrt{(l + m - 1)(l - m + 1)}, \\ c_{l-1, l, m+1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} &= \tilde{c}_{l-1, l, m+1; i', \dot{m}}^{k'k; \dot{k}'\dot{k}} \sqrt{(l + m + 1)(l - m - 1)}, \end{aligned}$$

we obtain

$$\begin{aligned} & [2 - (l + m + 1)(l - m) - (l + m - 1)(l - m)] \tilde{c}_{l-1,l,m;l',\dot{m}}^{k';k'k} \sqrt{(l + m)(l - m)} \\ & + (l + m - 1)(l - m + 1) \tilde{c}_{l-1,l,m-1;l',\dot{m}}^{k';k'k} \sqrt{(l - m)(l + m)} \\ & + (l - m - 1)(l + m + 1) \tilde{c}_{l-1,l,m+1;l',\dot{m}}^{k';k'k} \sqrt{(l + m)(l - m)} = 0. \end{aligned}$$

Where,

$$\begin{aligned} & 2[1 - l^2 + m^2] \tilde{c}_{l-1,l,m;l',\dot{m}}^{k';k'k} + [l^2 - (m - 1)^2] \tilde{c}_{l-1,l,m-1;l',\dot{m}}^{k';k'k} \\ & + [l^2 - (m + 1)^2] \tilde{c}_{l-1,l,m+1;l',\dot{m}}^{k';k'k} = 0. \end{aligned}$$

It is easy to verify that this system can be solved at $\tilde{c}_{l-1,l,m;l',\dot{m}}^{k';k'k}$ (this coefficient does not depend on m). For that reason, we can suppose $\tilde{c}_{l-1,l,m;l',\dot{m}}^{k';k'k} = c_{l-1,l;l',\dot{m}}^{k';k'k}$. Coming back to the old variables, we find that

$$c_{l-1,l,m;l',\dot{m}}^{k';k'k} = c_{l-1,l;l',\dot{m}}^{k';k'k} \sqrt{l^2 - m^2}.$$

Finally, let us suppose $l' = l + 1$. In this case, the system (71) takes the form

$$\begin{aligned} & [2 - (l + m + 1)(l - m) - (l + m + 1)(l - m + 2)] c_{l+1,l,m;l',\dot{m}}^{k';k'k} \\ & + \sqrt{(l + m + 1)(l - m + 2)(l + m)(l - m + 1)} c_{l+1,l,m-1;l',\dot{m}}^{k';k'k} \\ & + \sqrt{(l + m + 2)(l - m + 1)(l + m + 1)(l - m)} c_{l+1,l,m+1;l',\dot{m}}^{k';k'k} = 0. \end{aligned}$$

Making the substitutions

$$\begin{aligned} c_{l+1,l,m;l',\dot{m}}^{k';k'k} &= \sqrt{(l + m)(l - m + 1)} c_{l+1,l,m;l',\dot{m}}^{k';k'k}, \\ c_{l+1,l,m-1;l',\dot{m}}^{k';k'k} &= \sqrt{(l + m)(l - m + 2)} c_{l+1,l,m-1;l',\dot{m}}^{k';k'k}, \\ c_{l+1,l,m+1;l',\dot{m}}^{k';k'k} &= \sqrt{(l + m + 2)(l - m)} c_{l+1,l,m+1;l',\dot{m}}^{k';k'k}, \end{aligned}$$

we obtain

$$\begin{aligned} & [2 - (l + m + 1)(l - m) - (l + m + 1)(l - m + 2)] \\ & \sqrt{(l + m + 1)(l - m + 1)} c_{l+1,l,m;l',\dot{m}}^{k';k'k} \\ & + (l + m)(l - m + 2) \sqrt{(l + m + 1)(l - m + 1)} c_{l+1,l,m-1;l',\dot{m}}^{k';k'k} \\ & + (l + m + 2)(l - m) \sqrt{(l + m + 1)(l - m + 1)} c_{l+1,l,m-1;l',\dot{m}}^{k';k'k} = 0. \end{aligned}$$

Where,

$$2[m^2 - l^2 - 2l]\tilde{c}_{l+1,l,m;l,\dot{m}}^{k'k;k'k} + [l^2 - m^2 + 2l + 2m]\tilde{c}_{l+1,l,m-1;l,\dot{m}}^{k'k;k'k} + [l^2 - m^2 + 2l - 2m]\tilde{c}_{l+1,l,m+1;l,\dot{m}}^{k'k;k'k} = 0.$$

A solution $c_{l+1,l;l,\dot{m}}^{k'k;k'k}$ of the latter equation does not depend on m also.

Thus, the action of the commutator $[[\Lambda_3^{\dot{l}i}, X_-^{\dot{l}i}], X_+^{\dot{l}i}]$ on the basis vectors $\zeta_{lm;\dot{m}}^{kk}$ gives us the following solutions:

$$\begin{aligned} c_{l-1,l,m;l,\dot{m}}^{k'k;k'k} &= c_{l-1,l;l,\dot{m}}^{k'k;k'k} \sqrt{l^2 - m^2}, \\ c_{ll,m;l,\dot{m}}^{k'k;k'k} &= c_{ll;l,\dot{m}}^{k'k;k'k} m, \\ c_{l+1,l,m;l,\dot{m}}^{k'k;k'k} &= c_{l+1,l;l,\dot{m}}^{k'k;k'k} \sqrt{(l+1)^2 - m^2}. \end{aligned}$$

With the aim to find the final form for nonzero elements of $\Lambda_3^{\dot{l}i}$ we must apply the commutator $[[\Lambda_3^{\dot{l}i}, Y_-^{\dot{l}i}], Y_+^{\dot{l}i}]$. In the result, we have

$$\begin{aligned} &[[\Lambda_3^{\dot{l}i}, Y_-^{\dot{l}i}], Y_+^{\dot{l}i}]\zeta_{lm;\dot{m}}^{kk} \\ &= \sum_{l',k',i',k'} \{[(\alpha_{\dot{m}+1}^j)^2 + (\alpha_{\dot{m}}^i)^2]c_{l',l,m;l,\dot{m}}^{k'k;k'k} - \alpha_{\dot{m}}^i \alpha_{\dot{m}}^j c_{l',l,m;l,\dot{m}-1}^{k'k;k'k} - \alpha_{\dot{m}+1}^j \alpha_{\dot{m}+1}^i c_{l',l,m;l,\dot{m}+1}^{k'k;k'k}\} \zeta_{l'm;l,\dot{m}}^{k'k'}. \end{aligned}$$

Hence, it follows

$$\begin{aligned} 2c_{l',l,m;l,\dot{m}}^{k'k;k'k} &= [(\alpha_{\dot{m}+1}^j)^2 + (\alpha_{\dot{m}}^i)^2]c_{l',l,m;l,\dot{m}}^{k'k;k'k} - \alpha_{\dot{m}}^i \alpha_{\dot{m}}^j c_{l',l,m;l,\dot{m}-1}^{k'k;k'k} \\ &\quad - \alpha_{\dot{m}+1}^j \alpha_{\dot{m}+1}^i c_{l',l,m;l,\dot{m}+1}^{k'k;k'k}. \end{aligned}$$

Solutions of the latter system are derived by means of the analogous calculations presented in the previous case of the commutator $[[\Lambda_3^{\dot{l}i}, X_-^{\dot{l}i}], X_+^{\dot{l}i}]$. They have the form

$$\begin{aligned} c_{l',l,m;j-1,\dot{m}}^{k'k;k'k} &= c_{l',l,m;j-1,\dot{m}}^{k'k;k'k} \sqrt{l'^2 - \dot{m}^2}, \\ c_{l',l,m;j,\dot{m}}^{k'k;k'k} &= c_{l',l,m;j,\dot{m}}^{k'k;k'k} \dot{m}, \\ c_{l',l,m;j+1,\dot{m}}^{k'k;k'k} &= c_{l',l,m;j+1,\dot{m}}^{k'k;k'k} \sqrt{(j+1)^2 - \dot{m}^2}. \end{aligned}$$

Thus, matrix elements of Λ_3^{ij} are

$$\Lambda_3^{ij} : \begin{cases} c_{l-1,l,m;i-1,i,\dot{m}}^{k'k;k'k} = c_{l-1,l;i-1,i}^{k'k;k'k} \sqrt{(l^2 - m^2)(l^2 - \dot{m}^2)}, \\ c_{l,m;\dot{l},\dot{m}}^{k'k;k'k} = c_{l,\dot{l}}^{k'k;k'k} m\dot{m}, \\ c_{l+1,l,m;i+1,i,\dot{m}}^{k'k;k'k} = c_{l+1,l;i+1,i}^{k'k;k'k} \sqrt{((l+1)^2 - m^2)((l+1)^2 - \dot{m}^2)}. \end{cases} \quad (72)$$

All other elements of the matrix Λ_3^{ij} are equal to zero.

Let us define now elements of the matrices Λ_1^{ij} and Λ_2^{ij} . For the transformation Λ_1^{ij} in the helicity basis, we have

$$\Lambda_1^{ij} \zeta_{lm;\dot{l}\dot{m}}^{kk} = \sum_{l',m',\dot{l}',\dot{m}',k',k'} a_{l',m';\dot{l}',\dot{m}'}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'}.$$

We will find the numbers $a_{l',m';\dot{l}',\dot{m}'}^{k'k;k'k}$ using the relations $\Lambda_1^{ij} = [\Lambda_2^{ij}, \Lambda_3^{ij}]$ or $\Lambda_1^{ij} = i[\mathbf{B}_2^{ij}, \Lambda_3^{ij}]$ and (9) (or (12)). Indeed,

$$\begin{aligned} \Lambda_1^{ij} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \Lambda_2^{ij} \Lambda_3^{ij} \zeta_{lm;\dot{l}\dot{m}}^{kk} - \Lambda_3^{ij} \Lambda_2^{ij} \zeta_{lm;\dot{l}\dot{m}}^{kk} \\ &= \Lambda_2^{ij} \sum_{l',m',\dot{l}',\dot{m}',k',k'} c_{l',m';\dot{l}',\dot{m}'}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'} - \frac{1}{2} \Lambda_3^{ij} \left(\alpha_m^l \zeta_{l,m-1;j,\dot{m}}^{kk} - \alpha_{m+1}^l \zeta_{l,m+1;j\dot{m}}^{kk} \right. \\ &\quad \left. - \alpha_{\dot{m}}^j \zeta_{l,m;j,\dot{m}-1}^{kk} + \alpha_{\dot{m}+1}^j \zeta_{l,m;j,\dot{m}+1}^{kk} \right) = \frac{1}{2} \sum_{l',m',\dot{l}',\dot{m}',k',k'} c_{l',m';\dot{l}',\dot{m}'}^{k'k;k'k} \\ &\quad \times \left(\alpha_{m'}^l \zeta_{l',m'-1;j,\dot{m}'}^{k'k'} - \alpha_{m'+1}^l \zeta_{l',m'+1;j,\dot{m}'}^{k'k'} - \alpha_{\dot{m}'}^j \zeta_{l',m';\dot{l}',\dot{m}'-1}^{k'k'} + \alpha_{\dot{m}'+1}^j \zeta_{l',m';\dot{l}',\dot{m}'+1}^{k'k'} \right) \\ &\quad - \frac{1}{2} \alpha_m^l \sum_{l',m',\dot{l}',\dot{m}',k',k'} c_{l',m',m-1;\dot{l}',\dot{m}'}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'} + \frac{1}{2} \alpha_{m+1}^l \\ &\quad \times \sum_{l',m',\dot{l}',\dot{m}',k',k'} c_{l',m',m+1;\dot{l}',\dot{m}'}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'} \\ &\quad + \frac{1}{2} \alpha_{\dot{m}}^j \sum_{l',m',\dot{l}',\dot{m}',k',k'} c_{l',m',m;\dot{l}',\dot{m}'-1}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'} - \frac{1}{2} \alpha_{\dot{m}+1}^j \\ &\quad \times \sum_{l',m',\dot{l}',\dot{m}',k',k'} c_{l',m',m;\dot{l}',\dot{m}'+1}^{k'k;k'k} \zeta_{l'm';\dot{l}'\dot{m}'}^{k'k'}. \end{aligned}$$

Dividing the first sum on the four and changing the summation index in the each sums obtained, we come to the following expression:

$$\Lambda_1^{lj} \zeta_{lm;\dot{m}}^{kk} = \frac{1}{2} \sum_{l',m',\dot{l}',\dot{m}',k',\dot{k}'} \left(\alpha_{m'+1}^{l'} c_{l',m'+1,m;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} - \alpha_{m'}^{l'} c_{l',m'-1,m;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \right. \\ \left. - \alpha_{\dot{m}'+1}^{\dot{l}'} c_{l',m',m;\dot{l}',\dot{m}'+1,\dot{m}}^{k'k;k'\dot{k}} + \alpha_{\dot{m}}^{\dot{l}'} c_{l',m',m;\dot{l}',\dot{m}'-1,\dot{m}}^{k'k;k'\dot{k}} - \alpha_m^l c_{l',m',m-1;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \right. \\ \left. + \alpha_{m+1}^l c_{l',m',m+1;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} + \alpha_{\dot{m}}^{\dot{l}} c_{l',m',m;\dot{l}',\dot{m}'\dot{m}-1}^{k'k;k'\dot{k}} - \alpha_{\dot{m}+1}^{\dot{l}} c_{l',m',m;\dot{l}',\dot{m}'\dot{m}+1}^{k'k;k'\dot{k}} \right) \zeta_{l'm';\dot{l}'\dot{m}'}$$

Therefore, elements of the matrix $\Lambda_1^{\dot{l}l}$ have the form

$$a_{l',m',m;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \\ = \frac{1}{2} \left(\alpha_{m'+1}^{l'} c_{l',m'+1,m;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} - \alpha_{m'}^{l'} c_{l',m'-1,m;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} - \alpha_{\dot{m}'+1}^{\dot{l}'} c_{l',m',m;\dot{l}',\dot{m}'+1,\dot{m}}^{k'k;k'\dot{k}} \right. \\ \left. + \alpha_{\dot{m}}^{\dot{l}'} c_{l',m',m;\dot{l}',\dot{m}'-1,\dot{m}}^{k'k;k'\dot{k}} - \alpha_m^l c_{l',m',m-1;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} + \alpha_{m+1}^l c_{l',m',m+1;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \right. \\ \left. + \alpha_{\dot{m}}^{\dot{l}} c_{l',m',m;\dot{l}',\dot{m}'\dot{m}-1}^{k'k;k'\dot{k}} - \alpha_{\dot{m}+1}^{\dot{l}} c_{l',m',m;\dot{l}',\dot{m}'\dot{m}+1}^{k'k;k'\dot{k}} \right) \zeta_{l'm';\dot{l}'\dot{m}'}. \tag{73}$$

Since $c_{l',m',m;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \neq 0$ only at $m' = m, \dot{m}' = \dot{m}$ and $l' = l - 1, l, l + 1, \dot{l}' = \dot{l} - 1, \dot{l}, \dot{l} + 1$, then at the fixed indices $m, \dot{m}, l, \dot{l}, k', k, \dot{k}', \dot{k}$, we have twelve numbers $a_{l',m',m;\dot{l}',\dot{m}'\dot{m}}^{k'k;k'\dot{k}}$ which are different from zero. Substituting $\alpha_m^l = \sqrt{(l+m)(l-m+1)}$, $\alpha_{\dot{m}}^{\dot{l}} = \sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)}$ into (73) and using $c_{l'm';\dot{l}'\dot{m}'}^{k'k;k'\dot{k}}$ from (72), we find that

$$a_{l-1,l,m-1,m;\dot{l}-1,\dot{l},\dot{m}\dot{m}}^{k'k;k'\dot{k}} = -\frac{1}{2} c_{l-1,l;\dot{l}-1,\dot{l}}^{k'k;k'\dot{k}} \sqrt{(l+m)(l+m-1)(\dot{l}^2 - \dot{m}^2)}, \\ a_{ll,m-1,m;\dot{l}\dot{l},\dot{m}\dot{m}}^{k'k;k'\dot{k}} = \frac{1}{2} c_{ll;\dot{l}\dot{l}}^{k'k;k'\dot{k}} \dot{m} \sqrt{(l+m)(l-m+1)}, \\ a_{l+1,l,m-1,m;\dot{l}+1,\dot{l},\dot{m}\dot{m}}^{k'k;k'\dot{k}} = \frac{1}{2} c_{l+1,l;\dot{l}+1,\dot{l}}^{k'k;k'\dot{k}} \sqrt{(l-m+1)(l-m+2)((\dot{l}+1)^2 - \dot{m}^2)}, \\ a_{l-1,l,m+1,m;\dot{l}-1,\dot{l},\dot{m}\dot{m}}^{k'k;k'\dot{k}} = \frac{1}{2} c_{l-1,l;\dot{l}-1,\dot{l}}^{k'k;k'\dot{k}} \sqrt{(l-m)(l-m-1)(\dot{l}^2 - \dot{m}^2)}, \\ a_{ll,m+1,m;\dot{l}\dot{l},\dot{m}\dot{m}}^{k'k;k'\dot{k}} = \frac{1}{2} c_{ll;\dot{l}\dot{l}}^{k'k;k'\dot{k}} \dot{m} \sqrt{(l+m+1)(l-m)},$$

$$\begin{aligned}
 a_{l+1,l,m+1,m;j+1,i,\dot{m}\dot{m}}^{k'k;k'k} &= -\frac{1}{2}c_{l+1,l;j+1,i}^{k'k;k'k} \sqrt{(l+m+1)(l+m+2)((j+1)^2 - \dot{m}^2)}, \\
 a_{l-1,l,m,m;j-1,i,\dot{m}-1,\dot{m}}^{k'k;k'k} &= \frac{1}{2}c_{l-1,l;j-1,i}^{k'k;k'k} \sqrt{(l^2 - m^2)(j+\dot{m})(j+\dot{m}-1)}, \\
 a_{ll,m,m;\dot{l},\dot{m}-1,\dot{m}}^{k'k;k'k} &= -\frac{1}{2}c_{ll;\dot{l}}^{k'k;k'k} m \sqrt{(j+\dot{m})(j-\dot{m}+1)}, \\
 a_{l+1,l,m,m;j+1,i,\dot{m}-1,\dot{m}}^{k'k;k'k} &= -\frac{1}{2}c_{l+1,l;j+1,i}^{k'k;k'k} \sqrt{((l+1)^2 - m^2)(j-\dot{m}+1)(j-\dot{m}+2)}, \\
 a_{l-1,l,m,m;j-1,i,\dot{m}+1,\dot{m}}^{k'k;k'k} &= -\frac{1}{2}c_{l-1,l;j-1,i}^{k'k;k'k} \sqrt{(l^2 - m^2)(j-\dot{m})(j-\dot{m}-1)}, \\
 a_{ll,m,m;\dot{l},\dot{m}+1,\dot{m}}^{k'k;k'k} &= -\frac{1}{2}c_{ll;\dot{l}}^{k'k;k'k} m \sqrt{(j+\dot{m}+1)(j-\dot{m})}, \\
 a_{l+1,l,m,m;j+1,i,\dot{m}+1,\dot{m}}^{k'k;k'k} &= \frac{1}{2}c_{l+1,l;j+1,i}^{k'k;k'k} \sqrt{((l+1)^2 - m^2)(j+\dot{m}+1)(j+\dot{m}+2)}.
 \end{aligned}
 \tag{74}$$

Let us define elements of the matrix Λ_2^{ij} . From $\Lambda_2^{ij} = -[A_1^i, \Lambda_3^j]$ or $\Lambda_2^{ij} = -i[B_1^j, \Lambda_3^i]$

and
$$\Lambda_2^{ij} \zeta_{lm;\dot{m}}^{kk} = \sum_{l',m',\dot{l}',\dot{m}',k',k'} b_{l',m';\dot{l}',\dot{m}'}^{k'k;k'k} \zeta_{l'm';\dot{l}\dot{m}}^{k'k'}$$

and also the relations (8) (or (11)), (72) it follows that

$$\begin{aligned}
 b_{l-1,l,m-1,m;j-1,i,\dot{m}\dot{m}}^{k'k;k'k} &= -\frac{i}{2}c_{l-1,l;j-1,i}^{k'k;k'k} \sqrt{(l+m)(l+m-1)(\dot{l}^2 - \dot{m}^2)}, \\
 b_{ll,m-1,m;\dot{l},\dot{m}\dot{m}}^{k'k;k'k} &= \frac{i}{2}c_{ll;\dot{l}}^{k'k;k'k} \dot{m} \sqrt{(l+m)(l-m+1)}, \\
 b_{l+1,l,m-1,m;j+1,i,\dot{m}\dot{m}}^{k'k;k'k} &= \frac{i}{2}c_{l+1,l;j+1,i}^{k'k;k'k} \sqrt{(l-m+1)(l-m+2)((j+1)^2 - \dot{m}^2)}, \\
 b_{l-1,l,m+1,m;j-1,i,\dot{m}\dot{m}}^{k'k;k'k} &= -\frac{i}{2}c_{l-1,l;j-1,i}^{k'k;k'k} \sqrt{(l-m)(l-m-1)(\dot{l}^2 - \dot{m}^2)}, \\
 b_{ll,m+1,m;\dot{l},\dot{m}\dot{m}}^{k'k;k'k} &= -\frac{i}{2}c_{ll;\dot{l}}^{k'k;k'k} \dot{m} \sqrt{(l+m+1)(l-m)}, \\
 b_{l+1,l,m+1,m;j+1,i,\dot{m}\dot{m}}^{k'k;k'k} &= \frac{i}{2}c_{l+1,l;j+1,i}^{k'k;k'k} \sqrt{(l+m+1)(l+m+2)((j+1)^2 - \dot{m}^2)},
 \end{aligned}$$

$$\begin{aligned}
 b_{l-1,l,m,m;j-1,i,\dot{m}-1,\dot{m}}^{k'k;k'k} &= \frac{i}{2} c_{l-1,l;i-1,i}^{k'k;k'k} \sqrt{(l^2 - m^2)(j + \dot{m})(j + \dot{m} - 1)}, \\
 b_{ll,m,m;\dot{l},\dot{m}-1,\dot{m}}^{k'k;k'k} &= -\frac{i}{2} c_{ll;\dot{l}}^{k'k;k'k} m \sqrt{(j + \dot{m})(j - \dot{m} + 1)}, \\
 b_{l+1,l,m,m;j+1,i,\dot{m}-1,\dot{m}}^{k'k;k'k} &= -\frac{i}{2} c_{l+1,l;i+1,i}^{k'k;k'k} \sqrt{((l + 1)^2 - m^2)(j - \dot{m} + 1)(j - \dot{m} + 2)}, \\
 b_{l-1,l,m,m;j-1,i,\dot{m}+1,\dot{m}}^{k'k;k'k} &= \frac{i}{2} c_{l-1,l;i-1,i}^{k'k;k'k} \sqrt{(l^2 - m^2)(j - \dot{m})(j - \dot{m} - 1)}, \\
 b_{ll,m,m;\dot{l},\dot{m}+1,\dot{m}}^{k'k;k'k} &= \frac{i}{2} c_{ll;\dot{l}}^{k'k;k'k} m \sqrt{(j + \dot{m} + 1)(j - \dot{m})}, \\
 b_{l+1,l,m,m;j+1,i,\dot{m}+1,\dot{m}}^{k'k;k'k} &= -\frac{i}{2} c_{l+1,l;i+1,i}^{k'k;k'k} \sqrt{((l + 1)^2 - m^2)(j + \dot{m} + 1)(j + \dot{m} + 2)}.
 \end{aligned} \tag{75}$$

Coming to the dual representations, we find elements of the matrices Λ_1^{*li} , Λ_2^{*li} and Λ_3^{*li} . The dual transformations Λ_i^{*li} in the helicity basis are

$$\begin{aligned}
 \Lambda_1^{*li} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \sum_{l',m',i',\dot{m}',k',k'} a_{l',m',m';i',\dot{m}',\dot{m}}^{*k'k;k'k} \zeta_{lm;\dot{l}'\dot{m}'}^{k'k'}, \\
 \Lambda_2^{*li} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \sum_{l',m',i',\dot{m}',k',k'} b_{l',m',m';i',\dot{m}',\dot{m}}^{*k'k;k'k} \zeta_{lm;\dot{l}'\dot{m}'}^{k'k'}, \\
 \Lambda_3^{*li} \zeta_{lm;\dot{l}\dot{m}}^{kk} &= \sum_{l',m',i',\dot{m}',k',k'} c_{l',m',m';i',\dot{m}',\dot{m}}^{*k'k;k'k} \zeta_{lm;\dot{l}'\dot{m}'}^{k'k'}.
 \end{aligned}$$

Calculating the commutators $[*_3^d \Lambda_3^{li}, Y_3^{li}]$, $[[*_3^d \Lambda_3^{li}, Y_-^{li}], Y_+^{li}]$ with respect to the vectors $\zeta_{lm;\dot{l}\dot{m}}^{kk}$, we find elements of the matrix Λ_3^{*li} :

$$\Lambda_3^{*li} : \begin{cases} c_{l-1,l,m,i;j-1,i,\dot{m}}^{*k'k;k'k} = c_{l-1,l;i-1,i}^{*k'k;k'k} \sqrt{(l^2 - m^2)(l^2 - \dot{m}^2)}, \\ c_{ll,m,i;\dot{l},\dot{m}}^{*k'k;k'k} = c_{ll;\dot{l}}^{*k'k;k'k} m \dot{m}, \\ c_{l+1,l,m,i;j+1,i,\dot{m}}^{*k'k;k'k} = c_{l+1,l;i+1,i}^{*k'k;k'k} \sqrt{((l + 1)^2 - m^2)((l + 1)^2 - \dot{m}^2)}. \end{cases} \tag{76}$$

Using the relations $\Lambda_1^{*ii} = [\tilde{A}_2^{ii}, \Lambda_3]$ or $\Lambda_1^{*ii} = -i[\tilde{B}_2^{ii}, \Lambda_3]$ and (15) (or (18)), we find elements $a_{l',m';i',\dot{m}'\dot{m}}^{*k'k;k'k}$ of the matrix Λ_1^{*ii} :

$$\begin{aligned}
 a_{l-1,l,m-1,m;i-1,i,\dot{m}\dot{m}}^{*k'k;k'k} &= \frac{1}{2} c_{l-1,l;i-1,i} \sqrt{(l+m)(l+m-1)(i^2 - \dot{m}^2)}, \\
 a_{ll,m-1,m;i\dot{l},\dot{m}\dot{m}}^{*k'k;k'k} &= -\frac{1}{2} c_{ll;i\dot{l}} \dot{m} \sqrt{(l+m)(l-m+1)}, \\
 a_{l+1,l,m-1,m;i+1,i,\dot{m}\dot{m}}^{*k'k;k'k} &= -\frac{1}{2} c_{l+1,l;i+1,i} \sqrt{(l-m+1)(l-m+2)((i+1)^2 - \dot{m}^2)}, \\
 a_{l-1,l,m+1,m;i-1,i,\dot{m}\dot{m}}^{*k'k;k'k} &= -\frac{1}{2} c_{l-1,l;i-1,i} \sqrt{(l-m)(l-m-1)(i^2 - \dot{m}^2)}, \\
 a_{ll,m+1,m;i\dot{l},\dot{m}\dot{m}}^{*k'k;k'k} &= -\frac{1}{2} c_{ll;i\dot{l}} \dot{m} \sqrt{(l+m+1)(l-m)}, \\
 a_{l+1,l,m+1,m;i+1,i,\dot{m}\dot{m}}^{*k'k;k'k} &= \frac{1}{2} c_{l+1,l;i+1,i} \sqrt{(l+m+1)(l+m+2)((i+1)^2 - \dot{m}^2)}, \\
 a_{l-1,l,m,m;i-1,i,\dot{m}-1,\dot{m}}^{*k'k;k'k} &= -\frac{1}{2} c_{l-1,l;i-1,i} \sqrt{(l^2 - m^2)(i + \dot{m})(i + \dot{m} - 1)}, \\
 a_{ll,m,m;i\dot{l},\dot{m}-1,\dot{m}}^{*k'k;k'k} &= \frac{1}{2} c_{ll;i\dot{l}} m \sqrt{(i + \dot{m})(i - \dot{m} + 1)}, \\
 a_{l+1,l,m,m;i+1,i,\dot{m}-1,\dot{m}}^{*k'k;k'k} &= \frac{1}{2} c_{l+1,l;i+1,i} \sqrt{((l+1)^2 - m^2)(i - \dot{m} + 1)(i - \dot{m} + 2)}, \\
 a_{l-1,l,m,m;i-1,i,\dot{m}+1,\dot{m}}^{*k'k;k'k} &= \frac{1}{2} c_{l-1,l;i-1,i} \sqrt{(l^2 - m^2)(i - \dot{m})(i - \dot{m} - 1)}, \\
 a_{ll,m,m;i\dot{l},\dot{m}+1,\dot{m}}^{*k'k;k'k} &= \frac{1}{2} c_{ll;i\dot{l}} m \sqrt{(i + \dot{m} + 1)(i - \dot{m})}, \\
 a_{l+1,l,m,m;i+1,i,\dot{m}+1,\dot{m}}^{*k'k;k'k} &= -\frac{1}{2} c_{l+1,l;i+1,i} \sqrt{((l+1)^2 - m^2)(i + \dot{m} + 1)(i + \dot{m} + 2)}.
 \end{aligned}$$

(77)

Further, from the relations $\Lambda_2^{*i} = -[\tilde{A}_1^i, \Lambda_3^{*i}]$ or $\Lambda_2^{*i} = i[\tilde{B}_1^i, \Lambda_3^{*i}]$, we obtain elements $b_{l^p r i m \ell, m' m'; i', i, \dot{m} \dot{m}}^{*k' k; k' k}$ of Λ_2^{*i} . All calculations are analogous to the calculations presented for the case of Λ_i^i . In the result, we have

$$\begin{aligned}
 b_{l-1, l, m-1, m; i-1, i, \dot{m} \dot{m}}^{*k' k; k' k} &= \frac{i}{2} c_{l-1, l; i-1, i} \sqrt{(l+m)(l+m-1)(i^2 - \dot{m}^2)}, \\
 b_{ll, m-1, m; i, i, \dot{m} \dot{m}}^{*k' k; k' k} &= -\frac{i}{2} c_{ll; i} \dot{m} \sqrt{(l+m)(l-m+1)}, \\
 b_{l+1, l, m-1, m; i+1, i, \dot{m} \dot{m}}^{*k' k; k' k} &= -\frac{i}{2} c_{l+1, l; i+1, i} \sqrt{(l-m+1)(l-m+2)((i+1)^2 - \dot{m}^2)}, \\
 b_{l-1, l, m+1, m; i-1, i, \dot{m} \dot{m}}^{*k' k; k' k} &= \frac{i}{2} c_{l-1, l; i-1, i} \sqrt{(l-m)(l-m-1)(i^2 - \dot{m}^2)}, \\
 b_{ll, m+1, m; i, i, \dot{m} \dot{m}}^{*k' k; k' k} &= \frac{i}{2} c_{ll; i} \dot{m} \sqrt{(l+m+1)(l-m)}, \\
 b_{l+1, l, m+1, m; i+1, i, \dot{m} \dot{m}}^{*k' k; k' k} &= -\frac{i}{2} c_{l+1, l; i+1, i} \sqrt{(l+m+1)(l+m+2)((i+1)^2 - \dot{m}^2)}, \\
 b_{l-1, l, m, m; i-1, i, \dot{m}-1, \dot{m}}^{*k' k; k' k} &= -\frac{i}{2} c_{l-1, l; i-1, i} \sqrt{(l^2 - m^2)(i + \dot{m})(i + \dot{m} - 1)}, \\
 b_{ll, m, m; i, i, \dot{m}-1, \dot{m}}^{*k' k; k' k} &= \frac{i}{2} c_{ll; i} m \sqrt{(i + \dot{m})(i - \dot{m} + 1)}, \\
 b_{l+1, l, m, m; i+1, i, \dot{m}-1, \dot{m}}^{*k' k; k' k} &= \frac{i}{2} c_{l+1, l; i+1, i} \sqrt{((l+1)^2 - m^2)(i - \dot{m} + 1)(i - \dot{m} + 2)}, \\
 b_{l-1, l, m, m; i-1, i, \dot{m}+1, \dot{m}}^{*k' k; k' k} &= -\frac{i}{2} c_{l-1, l; i-1, i} \sqrt{(l^2 - m^2)(i - \dot{m})(i - \dot{m} - 1)}, \\
 b_{ll, m, m; i, i, \dot{m}+1, \dot{m}}^{*k' k; k' k} &= -\frac{i}{2} c_{ll; i} m \sqrt{(i + \dot{m} + 1)(i - \dot{m})}, \\
 b_{l+1, l, m, m; i+1, i, \dot{m}+1, \dot{m}}^{*k' k; k' k} &= \frac{i}{2} c_{l+1, l; i+1, i} \sqrt{((l+1)^2 - m^2)(i + \dot{m} + 1)(i + \dot{m} + 2)}.
 \end{aligned}$$

(78)

In general, the matrix Λ_3^{li} must be a reducible representation of the proper Lorentz group \mathfrak{G}_+ , and can always be written in the form

$$\Lambda_3^{li} = \begin{bmatrix} \Lambda_3^{l_1 i_1} & & & & \\ & \Lambda_3^{l_2 i_2} & & 0 & \\ & & \Lambda_3^{l_3 i_3} & & \\ & 0 & & \ddots & \\ & & & & \Lambda_3^{l_n i_n} \end{bmatrix} \tag{79}$$

where $\Lambda_3^{l_i i_i}$ is a spin block (the matrix $\Lambda_3^{*l_i i_i}$ has the same decompositions). It is obvious that the matrices $\Lambda_1^{li}, \Lambda_2^{li}$ and $\Lambda_1^{*l_i i_i}, \Lambda_2^{*l_i i_i}$ also admit the decompositions of the type (79) by definition. If the spin block $\Lambda_3^{l_i i_i}$ has nonnull roots, then the particle possesses the spin $s_i = |l_i - i_i|$. The spin block Λ_3^s in (79) consists of the elements $c_{\tau\tau'}^s$, where τ_{l_1, i_2} and $\tau_{l_1', i_2'}$ are interlocking irreducible representations of the Lorentz group, that is, such representations, for which $l_1' = l_1 \pm \frac{1}{2}, i_2' = i_2 \pm \frac{1}{2}$. At this point, the block Λ_3^s contains only the elements $c_{\tau\tau'}^s$ corresponding to such interlocking representations $\tau_{l_1, i_2}, \tau_{l_1', i_2'}$ which satisfy the conditions

$$|l_1 - i_2| \leq s \leq l_1 + i_2, \quad |l_1' - i_2'| \leq s \leq l_1' + i_2'.$$

The interlocking irreducible representations of the Lorentz group also called as Bhabha (1945) and Gel'fand and Yaglom chains (1948).

Corresponding to the decomposition (79), the wave function also decomposes into a direct sum of component wave functions which we write

$$\psi = \psi_{l_1 m_1; i_1 \dot{m}_1} + \psi_{l_2 m_2; i_2 \dot{m}_2} + \psi_{l_3 m_3; i_3 \dot{m}_3} + \dots$$

According to a de Broglie theory of fusion (de Broglie, 1943), interlocking representations give rise to indecomposable RWE. Otherwise, we have decomposable equations. As is known, the indecomposable RWE correspond to composite particles. A relation between indecomposable RWE and composite particles will be studied in a separate work.

6. SEPARATION OF VARIABLES IN RWE

6.1. Boundary Value Problem

Following to the classical methods of mathematical physics (Courant and Hilbert, 1931), it is quite natural to set up a *boundary value problem for the relativistic wave equations (relativistically invariant system)*. It is well known that

all the physically meaningful requirements, which follow from the experience, are contained in the boundary value problem.

Let us construct in \mathbb{C}^3 the two-dimensional complex sphere \mathbb{S}^2 from the quantities $z_k = x_k + iy_k$, $z_k^* = x_k - iy_k$ as follows

$$\mathbf{z}^2 = z_1^2 + z_2^2 + z_3^2 = \mathbf{x}^2 - \mathbf{y}^2 + 2i\mathbf{xy} = r^2 \tag{80}$$

and its complex conjugate (dual) sphere \mathbb{S}^{*2} ,

$$\mathbf{z}^{*2} = z_1^{*2} + z_2^{*2} + z_3^{*2} = \mathbf{x}^2 - \mathbf{y}^2 - 2i\mathbf{xy} = r^{*2}. \tag{81}$$

For more details about the two-dimensional complex sphere (see Huszar, 1970a,b; Smorodinsky, 1970; Smorodinsky and Huszar, 1970). It is well-known that both quantities $\mathbf{x}^2 - \mathbf{y}^2$, \mathbf{xy} are invariant with respect to the Lorentz transformations, since a surface of the complex sphere is invariant (Casimir operators of the Lorentz group are constructed from such quantities, see also (20)). Moreover, since the real and imaginary parts of the complex two-sphere transform like the electric and magnetic fields, respectively, the invariance of $\mathbf{z}^2 \sim (\mathbf{E} + i\mathbf{B})^2$ under proper Lorentz transformations is evident. At this point, the quantities $\mathbf{x}^2 - \mathbf{y}^2$, \mathbf{xy} are similar to the well-known electromagnetic invariants $E^2 - B^2$, \mathbf{EB} . This intriguing relationship between the Laplace–Beltrami operators (20), Casimir operators of the Lorentz group and electromagnetic invariants $E^2 - B^2 \sim \mathbf{x}^2 - \mathbf{y}^2$, $\mathbf{EB} \sim \mathbf{xy}$ leads naturally to a Riemann–Silberstein representation of the electromagnetic field (see, for example, Weber, 1901; Silberstein, 1907; Bialynicki-Birula, 1996). In other words, the two-dimensional sphere, considered as a homogeneous space of the Poincaré group, is the most suitable arena for the subsequent investigations in quantum electrodynamics.

We will set up a boundary value problem for the two-dimensional complex sphere \mathbb{S}^2 (this problem can be considered as a relativistic generalization of the classical Dirichlet problem for the sphere S^2).

Let T be an unbounded region in $\mathbb{C}^3 \simeq \mathbb{R}^6$ and let Σ be a surface of the complex two-sphere (correspondingly, \mathbb{S}^* , for the dual two-sphere), then it needs to find a function $\psi(\mathfrak{g}) = (\psi_{m\dot{m}}(\mathfrak{g}), \dot{\psi}_{m\dot{m}}(\mathfrak{g}))^T$ satisfying the following conditions:

- (1) $\psi(\mathfrak{g})$ is a solution of the system

$$\sum_{j=1}^3 \Lambda_j^{li} \frac{\partial \psi}{\partial a_j} - i \sum_{j=1}^3 \Lambda_j^{li} \frac{\partial \psi}{\partial a_j^*} + \kappa^c \dot{\psi} = 0,$$

$$\sum_{j=1}^3 \Lambda_j^{*li} \frac{\partial \dot{\psi}}{\partial \bar{a}_j} + i \sum_{j=1}^3 \Lambda_j^{*li} \frac{\partial \dot{\psi}}{\partial \bar{a}_j^*} + \kappa^c \psi = 0,$$

in the all region T ;

- (2) $\psi(\mathbf{g})$ is a continuous function (everywhere in T), including the surfaces Σ and $\dot{\Sigma}$;
- (3) $\psi_{m\dot{m}}(\mathbf{g})|_{\Sigma} = F_{m\dot{m}}(\mathbf{g})$, $\dot{\psi}_{m\dot{m}}(\mathbf{g})|_{\dot{\Sigma}} = \dot{F}_{m\dot{m}}(\mathbf{g})$, where $F_{m\dot{m}}(\mathbf{g})$ and $\dot{F}_{m\dot{m}}(\mathbf{g})$ are square integrable functions defined on the surfaces Σ and $\dot{\Sigma}$, respectively.

In particular, boundary conditions can be represented by constants,

$$\psi(\mathbf{g})|_{\Sigma} = \text{const} = F_0, \quad \dot{\psi}(\mathbf{g})|_{\dot{\Sigma}} = \text{const} = \dot{F}_0.$$

It is obvious that an explicit form of the boundary conditions follows from the experience. For example, they can describe a distribution of energy in the experiment.

With the aim to solve the boundary value problem, we come to the complex Euler angles (1) and represent the function $\psi(r, \theta^c, \varphi^c) = (\psi_{m\dot{m}}(r, \theta^c, \varphi^c), \dot{\psi}_{m\dot{m}}(r^*, \dot{\theta}^c, \dot{\varphi}^c))^T$ in the form of following series:

$$\psi_{m\dot{m}}(r, \theta^c, \varphi^c) = \sum_{l,i=0}^{\infty} \sum_{k,\dot{k}} f_{l m k ; i \dot{m} \dot{k}}(r) \sum_{n=-l}^l \sum_{\dot{n}=-i}^i \alpha_{l n ; i \dot{n}}^{m \dot{m}} \mathfrak{M}_{m n ; \dot{m} \dot{n}}^{l i}(\varphi, \epsilon, \theta, \tau, 0, 0), \tag{82}$$

$$\dot{\psi}_{m\dot{m}}(r^*, \dot{\theta}^c, \dot{\varphi}^c) = \sum_{l,i=0}^{\infty} \sum_{k,\dot{k}} f_{l m k ; i \dot{m} \dot{k}}^*(r^*) \sum_{n=-l}^l \sum_{\dot{n}=-i}^i \alpha_{l n ; i \dot{n}}^{m \dot{m} *} \mathfrak{M}_{m n ; \dot{m} \dot{n}}^{* l i}(\varphi, \epsilon, \theta, \tau, 0, 0), \tag{83}$$

where

$$\begin{aligned} \alpha_{l n ; i \dot{n}}^{m \dot{m}} &= \frac{(-1)^n (2l + 1)(2i + 1)}{32\pi^4} \int_{\mathbb{S}^2} F_{m\dot{m}}(\theta^c, \varphi^c) \mathfrak{M}_{m n ; \dot{m} \dot{n}}^{l i}(\varphi, \epsilon, \theta, \tau, 0, 0) \sin \theta^c \\ &\quad \times \sin \dot{\theta}^c d\theta d\varphi d\tau d\epsilon, \\ \alpha_{l n ; i \dot{n}}^{m \dot{m} *} &= \frac{(-1)^{\dot{n}} (2l + 1)(2i + 1)}{32\pi^4} \int_{\mathbb{S}^2} \dot{F}_{m\dot{m}}(\dot{\theta}^c, \dot{\varphi}^c) \mathfrak{M}_{m n ; \dot{m} \dot{n}}^{* l i}(\varphi, \epsilon, \theta, \tau, 0, 0) \sin \theta^c \\ &\quad \times \sin \dot{\theta}^c d\theta d\varphi d\tau d\epsilon, \end{aligned}$$

The indices k and \dot{k} numerate equivalent representations. $\mathfrak{M}_{m n ; \dot{m} \dot{n}}^{l i}(\varphi, \epsilon, \theta, \tau, 0, 0)$ ($\mathfrak{M}_{m n ; \dot{m} \dot{n}}^{* l i}(\varphi, \epsilon, \theta, \tau, 0, 0)$) are hyperspherical functions defined on the surface Σ ($\dot{\Sigma}$) of the two-dimensional complex sphere of the radius r (r^*), $f_{l m k ; i \dot{m} \dot{k}}(r)$ and $f_{l m k ; i \dot{m} \dot{k}}^*(r^*)$ are radial functions. It is easy to see that we come here to the

harmonic analysis on the complex two-sphere, since the series (82) and (83) have the structure of the Fourier series on \mathbb{S}^2 .

Let us introduce now hyperspherical coordinates on the surfaces of the complex and dual spheres,

$$\begin{aligned} z_1 &= r \sin \theta^c \cos \varphi^c, & z_1^* &= r^* \sin \dot{\theta}^c \cos \dot{\varphi}^c, \\ z_2 &= r \sin \theta^c \sin \varphi^c, & z_2^* &= r^* \sin \dot{\theta}^c \sin \dot{\varphi}^c, \\ z_3 &= r \cos \theta^c, & z_3^* &= r^* \cos \dot{\theta}^c, \end{aligned} \tag{84}$$

where θ^c, φ^c are the complex Euler angles. Let us show that solutions of the equations (58) can be found in theorem of expansions in generalized hyperspherical functions considered in the previous section.

With this end in view, let us transform system (58) as follows. First of all, let us define the derivatives $\frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_i^*}$ on the surface of the two-dimensional complex sphere (80) and write them in the hyperspherical coordinates (84) as

$$\frac{\partial}{\partial a_1} = -\frac{\sin \varphi^c}{r \sin \theta^c} \frac{\partial}{\partial \varphi} + \frac{\cos \varphi^c \cos \theta^c}{r} \frac{\partial}{\partial \theta} + \cos \varphi^c \sin \theta^c \frac{\partial}{\partial r}, \tag{85}$$

$$\frac{\partial}{\partial a_2} = \frac{\cos \varphi^c}{r \sin \theta^c} \frac{\partial}{\partial \varphi} + \frac{\sin \varphi^c \cos \theta^c}{r} \frac{\partial}{\partial \theta} + \sin \varphi^c \sin \theta^c \frac{\partial}{\partial r}, \tag{86}$$

$$\frac{\partial}{\partial a_3} = -\frac{\sin \theta^c}{r} \frac{\partial}{\partial \theta} + \cos \theta^c \frac{\partial}{\partial r}. \tag{87}$$

$$\frac{\partial}{\partial a_1^*} = i \frac{\partial}{\partial a_1} = -\frac{\sin \dot{\varphi}^c}{r \sin \dot{\theta}^c} \frac{\partial}{\partial \dot{\varphi}} + \frac{\cos \dot{\varphi}^c \sin \dot{\theta}^c}{r} \frac{\partial}{\partial \dot{\theta}} + i \cos \dot{\varphi}^c \sin \dot{\theta}^c \frac{\partial}{\partial r^*}, \tag{88}$$

$$\frac{\partial}{\partial a_2^*} = i \frac{\partial}{\partial a_2} = \frac{\cos \dot{\varphi}^c}{r \sin \dot{\theta}^c} \frac{\partial}{\partial \dot{\varphi}} + \frac{\sin \dot{\varphi}^c \cos \dot{\theta}^c}{r} \frac{\partial}{\partial \dot{\theta}} + i \sin \dot{\varphi}^c \sin \dot{\theta}^c \frac{\partial}{\partial r^*}, \tag{89}$$

$$\frac{\partial}{\partial a_3^*} = i \frac{\partial}{\partial a_3} = -\frac{\sin \dot{\theta}^c}{r} \frac{\partial}{\partial \dot{\theta}} + i \cos \dot{\theta}^c \frac{\partial}{\partial r^*}. \tag{90}$$

Analogously, on the surface of the dual sphere (81), we have

$$\frac{\partial}{\partial \tilde{a}_1} = -\frac{\sin \varphi^c}{r^* \sin \theta^c} \frac{\partial}{\partial \varphi} + \frac{\cos \varphi^c \cos \theta^c}{r^*} \frac{\partial}{\partial \theta} + \cos \varphi^c \sin \theta^c \frac{\partial}{\partial r^*}, \tag{91}$$

$$\frac{\partial}{\partial \tilde{a}_2} = \frac{\cos \varphi^c}{r^* \sin \theta^c} \frac{\partial}{\partial \varphi} + \frac{\sin \varphi^c \cos \theta^c}{r^*} \frac{\partial}{\partial \theta} + \sin \varphi^c \sin \theta^c \frac{\partial}{\partial r^*}, \tag{92}$$

$$\frac{\partial}{\partial \tilde{a}_3} = -\frac{\sin \theta^c}{r^*} \frac{\partial}{\partial \theta} + \cos \theta^c \frac{\partial}{\partial r^*}. \tag{93}$$

$$\frac{\partial}{\partial \tilde{a}_1^*} = -i \frac{\partial}{\partial \tilde{a}_1} = \frac{\sin \dot{\varphi}^c}{r^* \sin \dot{\theta}^c} \frac{\partial}{\partial \dot{\varphi}} - \frac{\cos \dot{\varphi}^c \cos \dot{\theta}^c}{r^*} \frac{\partial}{\partial \dot{\theta}} - i \cos \dot{\varphi}^c \sin \dot{\theta}^c \frac{\partial}{\partial r^*}, \tag{94}$$

$$\frac{\partial}{\partial \tilde{a}_2^*} = -i \frac{\partial}{\partial \tilde{a}_2} = -\frac{\cos \varphi^c}{r^* \sin \theta^c} \frac{\partial}{\partial \epsilon} - \frac{\sin \varphi^c \cos \theta^c}{r^*} \frac{\partial}{\partial \tau} - i \sin \varphi^c \sin \theta^c \frac{\partial}{\partial r^*}, \quad (95)$$

$$\frac{\partial}{\partial \tilde{a}_3^*} = -i \frac{\partial}{\partial \tilde{a}_3} = \frac{\sin \theta^c}{r^*} \frac{\partial}{\partial \tau} - i \cos \theta^c \frac{\partial}{\partial r^*}. \quad (96)$$

Coming back to (58), we see that the matrices Λ_j^l and Λ_j^{*l} inherit their tensor structures from the infinitesimal operators (7),

$$\Lambda_j^l = \Lambda_j^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_j^l,$$

$$\Lambda_j^{*l} = \Lambda_j^{*l} \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_j^{*l}.$$

Taking into account the latter expressions, we rewrite the system (58) as follows

$$\begin{aligned} & \sum_{j=1}^3 \left(\Lambda_j^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_j^l \right) \frac{\partial \psi}{\partial a_j} \\ & + i \sum_{j=1}^3 \left(\Lambda_j^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_j^l \right) \frac{\partial \psi}{\partial a_j^*} + \kappa^c \psi = 0, \\ & \sum_{j=1}^3 \left(\Lambda_j^{*l} \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_j^{*l} \right) \frac{\dot{\psi}}{\partial \tilde{a}_j} \\ & - i \sum_{j=1}^3 \left(\Lambda_j^{*l} \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_j^{*l} \right) \frac{\dot{\psi}}{\partial \tilde{a}_j^*} + \kappa^c \dot{\psi} = 0. \end{aligned}$$

Substituting the functions $\psi = T_{li}^{-1}(\mathbf{g})\psi'$ ($\dot{\psi} = T_{li}^{*-1}(\mathbf{g})\dot{\psi}'$) and the derivatives (85)–(90), (91)–(96) into this system, and multiply by $T_{li}(\mathbf{g}) = T_{li}(\varphi^c, \theta^c, 0)$ ($T_{li}^*(\mathbf{g}) = T_{li}^*(\varphi^c, \theta^c, 0)$) from the left, we obtain

$$\begin{aligned} & T_{li}(\mathbf{g}) \left(\Lambda_1^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_1^l \right) \left[-\frac{\sin \varphi^c}{r \sin \theta^c} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \varphi} \right. \\ & \left. + \frac{\cos \varphi^c \cos \theta^c}{r} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \theta} + \cos \varphi^c \sin \theta^c \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial r} \right] \\ & + T_{li}(\mathbf{g}) \left(\Lambda_2^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_2^l \right) \left[\frac{\cos \varphi^c}{r \sin \theta^c} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \varphi} \right. \end{aligned}$$

$$\begin{aligned}
 & + \left. \frac{\sin \varphi^c \cos \theta^c}{r} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \theta} + \sin \varphi^c \sin \theta^c \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial r} \right] \\
 & + T_{li}(\mathbf{g})\Lambda_3^{li} \left[-\frac{\sin \theta^c}{r} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \theta} + \cos \theta^c \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial r} \right] \\
 & + iT_{li}(\mathbf{g})(\Lambda_1^l \otimes \mathbf{1}_{2i+1} - \mathbf{1}_{2l+1} \otimes \Lambda_1^i) \left[-\frac{\sin \dot{\varphi}^c}{r \sin \dot{\theta}^c} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \epsilon} \right. \\
 & \left. + \frac{\cos \dot{\varphi}^c \sin \dot{\theta}^c}{r} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \tau} + i \cos \dot{\varphi}^c \sin \dot{\theta}^c \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial r} \right] \\
 & + iT_{li}(\mathbf{g})(\Lambda_2^l \otimes \mathbf{1}_{2i+1} - \mathbf{1}_{2l+1} \otimes \Lambda_2^i) \left[-\frac{\cos \dot{\varphi}^c}{r \sin \dot{\theta}^c} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \epsilon} \right. \\
 & \left. + \frac{\sin \dot{\varphi}^c \cos \dot{\theta}^c}{r} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \theta} + i \sin \dot{\varphi}^c \sin \dot{\theta}^c \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial r} \right] \\
 & + iT_{li}(\mathbf{g})\Lambda_3^{li} \left[-\frac{\sin \dot{\varphi}^c}{r} \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial \tau} + i \cos \dot{\theta}^c \frac{\partial (T_{li}^{-1}(\mathbf{g})\psi')}{\partial r} \right] + \kappa^c \psi = 0, \\
 & {}^*T_{li}(\mathbf{g})(\Lambda_1^* \otimes \mathbf{1}_{2i+1} - \mathbf{1}_{2l+1} \otimes \Lambda_1^*) \left[-\frac{\sin \varphi^c}{r^* \sin \theta^c} \frac{\partial (T_{li}^{*-1}(\mathbf{g})\psi')}{\partial \varphi} \right. \\
 & \left. + \frac{\cos \varphi^c \cos \theta^c}{r^*} \frac{\partial (T_{li}^{*-1}(\mathbf{g})\psi')}{\partial \theta} + \cos \varphi^c \sin \theta^c \frac{\partial (T_{li}^{*-1}(\mathbf{g})\psi')}{\partial r^*} \right] \\
 & + {}^*T_{li}(\mathbf{g})(\Lambda_2^* \otimes \mathbf{1}_{2i+1} - \mathbf{1}_{2l+1} \otimes \Lambda_2^*) \left[\frac{\cos \varphi^c}{r^* \sin \theta^c} \frac{\partial (T_{li}^{*-1}(\mathbf{g})\psi')}{\partial \varphi} \right. \\
 & \left. + \frac{\sin \varphi^c \cos \theta^c}{r^*} \frac{\partial (T_{li}^{*-1}(\mathbf{g})\psi')}{\partial \theta} + \sin \varphi^c \sin \theta^c \frac{\partial (T_{li}^{*-1}(\mathbf{g})\psi')}{\partial r^*} \right] \\
 & + {}^*T_{li}(\mathbf{g})\Lambda_3^{*li} \left[-\frac{\sin \theta^c}{r^*} \frac{\partial (T_{li}^{*-1}(\mathbf{g})\psi')}{\partial \theta} + \cos \theta^c \frac{\partial (T_{li}^{*-1}(\mathbf{g})\psi')}{\partial r^*} \right]
 \end{aligned}$$

$$\begin{aligned}
& -iT_{li}^*(\mathfrak{g})(\Lambda_1^* \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_1^*) \left[\frac{\sin \varphi^c}{r^* \sin \theta^c} \frac{(T_{li}^{*-1}(\mathfrak{g})\psi')}{\partial \epsilon} \right. \\
& \left. - \frac{\cos \varphi^c \cos \theta^c}{r^*} \frac{(T_{li}^{*-1}(\mathfrak{g})\psi')}{\partial \tau} - i \cos \varphi^c \sin \theta^c \frac{(T_{li}^{*-1}(\mathfrak{g})\psi')}{\partial r^*} \right] \\
& -iT_{li}^*(\mathfrak{g})(\Lambda_2^* \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \Lambda_2^*) \left[-\frac{\cos \varphi^c}{r^* \sin \theta^c} \frac{(T_{li}^{*-1}(\mathfrak{g})\psi')}{\partial \epsilon} \right. \\
& \left. - \frac{\sin \varphi^c \cos \theta^c}{r^*} \frac{(T_{li}^{*-1}(\mathfrak{g})\psi')}{\partial \tau} - i \sin \varphi^c \sin \theta^c \frac{(T_{li}^{*-1}(\mathfrak{g})\psi')}{\partial r^*} \right] \\
& -iT_{li}^*(\mathfrak{g})\Lambda_3^* \left[\frac{\sin \theta^c}{r^*} \frac{(T_{li}^{*-1}(\mathfrak{g})\psi')}{\partial \tau} - i \cos \theta^c \frac{(T_{li}^{*-1}(\mathfrak{g})\psi')}{\partial r^*} \right] + \kappa^c \psi = 0.
\end{aligned}$$

In virtue of the invariance conditions (61) we have

$$\begin{aligned}
T_{li}(\mathfrak{g})[-\Lambda_1^{li} \sin \varphi^c + \Lambda_2^{li} \cos \varphi^c]T_{li}^{-1}(\mathfrak{g}) &= \Lambda_1^{li}, \\
T_{li}(\mathfrak{g})[\Lambda_1^{li} \cos \varphi^c \cos \theta^c + \Lambda_2^{li} \sin \varphi^c \cos \theta^c - \Lambda_3^{li} \sin \theta^c]T_{li}^{-1}(\mathfrak{g}) &= \Lambda_2, \\
T_{li}(\mathfrak{g})[2\Lambda_1^{li} \cos \varphi^c \sin \theta^c + \Lambda_2^{li} \sin \varphi^c \sin \theta^c + \Lambda_3^{li} \cos \theta^c]T_{li}^{-1}(\mathfrak{g}) &= \Lambda_3^{li}, \\
T_{li}(\mathfrak{g})[-\Lambda_1^{li} \sin \varphi^c + \Lambda_2^{li} \cos \varphi^c]T_{li}^{-1}(\mathfrak{g}) &= \Lambda_1^{li}, \\
T_{li}(\mathfrak{g})[\Lambda_1^{li} \cos \varphi^c \cos \theta^c + \Lambda_2^{li} \sin \varphi^c \cos \theta^c - \Lambda_3^{li} \sin \theta^c]T_{li}^{-1}(\mathfrak{g}) &= \Lambda_2^{li}, \\
T_{li}(\mathfrak{g})[\Lambda_1^{li} \cos \varphi^c \sin \theta^c + \Lambda_2^{li} \sin \varphi^c \sin \theta^c + \Lambda_3^{li} \cos \theta^c]T_{li}^{-1}(\mathfrak{g}) &= \Lambda_3^{li}, \\
T_{li}^*(\mathfrak{g})[-\Lambda_1^* \sin \varphi^c + \Lambda_2^* \cos \varphi^c]T_{li}^{*-1}(\mathfrak{g}) &= \Lambda_1^*, \\
T_{li}^*(\mathfrak{g})[\Lambda_1^* \cos \varphi^c \cos \theta^c + \Lambda_2^* \sin \varphi^c \cos \theta^c - \Lambda_3^* \sin \theta^c]T_{li}^{*-1}(\mathfrak{g}) &= \Lambda_2^*, \\
T_{li}^*(\mathfrak{g})[\Lambda_1^* \cos \varphi^c \sin \theta^c + \Lambda_2^* \sin \varphi^c \sin \theta^c + \Lambda_3^* \cos \theta^c]T_{li}^{*-1}(\mathfrak{g}) &= \Lambda_3^*, \\
T_{li}^*(\mathfrak{g})[\Lambda_1^* \sin \varphi^c - \Lambda_2^* \cos \varphi^c]T_{li}^{*-1}(\mathfrak{g}) &= \Lambda_1^*,
\end{aligned}$$

$$T_{li}^*(\mathbf{g})[-\Lambda_1^* \cos \varphi^c \cos \theta^c - \Lambda_2^* \sin \varphi^c \cos \theta^c + \Lambda_3^* \sin \theta^c] T_{li}^{*-1}(\mathbf{g}) = \Lambda_2^*,$$

$$T_{li}^*(\mathbf{g})[\Lambda_1^* \cos \varphi^c \sin \theta^c + \Lambda_2^* \sin \varphi^c \sin \theta^c + \Lambda_3^* \cos \theta^c] T_{li}^{*-1}(\mathbf{g}) = \Lambda_3^*.$$

Taking into account the latter relations, we can write system (58) as follows

$$\begin{aligned} & \frac{1}{r \sin \theta^c} \Lambda_1^l \otimes \mathbf{1}_{2i+1} T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial \varphi} - \frac{1}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes \Lambda_1^i T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial \varphi} \\ & + \frac{i}{r \sin \theta^c} \Lambda_1^l \otimes \mathbf{1}_{2i+1} T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial \epsilon} + \frac{i}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes \Lambda_1^i T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial \epsilon} \\ & - \frac{1}{r} \Lambda_2^l \otimes \mathbf{1}_{2i+1} T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial \theta} + \frac{1}{r} \mathbf{1}_{2l+1} \otimes \Lambda_2^i T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial \theta} \\ & - \frac{i}{r} \Lambda_2^l \otimes \mathbf{1}_{2i+1} T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial \tau} - \frac{i}{r} \mathbf{1}_{2l+1} \otimes \Lambda_2^i T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial \tau} \\ & - \Lambda_3^l T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial r} + i \Lambda_3^i T_{li}(\mathbf{g}) \frac{\partial (T_{li}^{-1}(\mathbf{g}) \psi')}{\partial r} + \kappa^c \psi' = 0, \\ & \frac{1}{r^* \sin \theta^c} \Lambda_1^* \otimes \mathbf{1}_{2l+1} T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial \varphi} - \frac{1}{r^* \sin \theta^c} \mathbf{1}_{2i+1} \otimes \Lambda_1^l T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial \varphi} \\ & + \frac{i}{r^* \sin \theta^c} \Lambda_1^* \otimes \mathbf{1}_{2l+1} T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial \epsilon} + \frac{i}{r^* \sin \theta^c} \mathbf{1}_{2i+1} \otimes \Lambda_1^l T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial \epsilon} \\ & - \frac{1}{r^*} \Lambda_2^* \otimes \mathbf{1}_{2l+1} T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial \theta} + \frac{1}{r^*} \mathbf{1}_{2i+1} \otimes \Lambda_2^l T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial \theta} \\ & - \frac{i}{r^*} \Lambda_2^* \otimes \mathbf{1}_{2l+1} T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial \tau} - \frac{i}{r^*} \mathbf{1}_{2i+1} \otimes \Lambda_2^l T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial \tau} \\ & + \Lambda_3^* T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial r^*} - i \Lambda_3^l T_{li}^*(\mathbf{g}) \frac{\partial (T_{li}^{*-1}(\mathbf{g}) \psi')}{\partial r^*} + \kappa^c \psi' = 0. \end{aligned} \quad (97)$$

The matrices $T_{li}^{-1}(\mathbf{g})$, $T_{li}^{*-1}(\mathbf{g})$ depend on φ , ϵ , θ , τ . Therefore, we must differentiate in $T_{li}^{-1}(\mathbf{g}) \psi'$ ($T_{li}^{*-1}(\mathbf{g}) \psi'$) the both factors. After differentiation we come to the following system:

$$\begin{aligned} & \frac{1}{r \sin \theta^c} \Lambda_1^l \otimes \mathbf{1}_{2i+1} \frac{\partial \psi'}{\partial \varphi} - \frac{1}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes \Lambda_1^i \frac{\partial \psi'}{\partial \varphi} + \frac{i}{r \sin \theta^c} \Lambda_1^l \otimes \mathbf{1}_{2i+1} \frac{\partial \psi'}{\partial \epsilon} \\ & - \frac{i}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes \Lambda_1^i \frac{\partial \psi'}{\partial \epsilon} - \frac{1}{r} \Lambda_2^l \otimes \mathbf{1}_{2i+1} \frac{\partial \psi'}{\partial \theta} + \frac{1}{r} \mathbf{1}_{2l+1} \otimes \Lambda_2^i \frac{\partial \psi'}{\partial \theta} \end{aligned}$$

$$\begin{aligned}
& -\frac{i}{r}\Lambda_2^l \otimes \mathbf{1}_{2i+1} \frac{\partial \psi'}{\partial \tau} - \frac{i}{r}\mathbf{1}_{2l+1} \otimes \Lambda_2^i \frac{\partial \psi'}{\partial \tau} + \Lambda_3^{ij} \frac{\partial \psi'}{\partial r} + i\Lambda_3^{li} \frac{\partial \psi'}{\partial r} \\
& + \left[\frac{1}{r \sin \theta^c} \Lambda_1^l \otimes \mathbf{1}_{2i+1} T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \varphi} - \frac{1}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes \Lambda_1^i T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \varphi} \right. \\
& + \frac{i}{r \sin \theta^c} \Lambda_1^l \otimes \mathbf{1}_{2i+1} T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \epsilon} - \frac{i}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes \Lambda_1^i T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \epsilon} \\
& - \frac{1}{r} \Lambda_2^l \otimes \mathbf{1}_{2i+1} T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \theta} + \frac{1}{r} \mathbf{1}_{2l+1} \otimes \Lambda_2^i T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \theta} \\
& \left. - \frac{i}{r} \Lambda_2^l \otimes \mathbf{1}_{2i+1} T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \tau} - \frac{i}{r} \mathbf{1}_{2l+1} \otimes \Lambda_2^i T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \tau} + \kappa^c I \right] \psi' = 0, \\
& \frac{1}{r^* \sin \theta^c} \Lambda_1^* \otimes \mathbf{1}_{2l+1} \frac{\partial \psi'}{\partial \varphi} - \frac{1}{r^* \sin \theta^c} \mathbf{1}_{2i+1} \otimes \Lambda_1^* \frac{\partial \psi'}{\partial \varphi} + \frac{i}{r^* \sin \theta^c} \Lambda_1^* \otimes \mathbf{1}_{2l+1} \frac{\partial \psi'}{\partial \epsilon} \\
& + \frac{i}{r^* \sin \theta^c} \mathbf{1}_{2i+1} \otimes \Lambda_1^* \frac{\partial \psi'}{\partial \epsilon} - \frac{1}{r^*} \Lambda_2^* \otimes \mathbf{1}_{2l+1} \frac{\partial \psi'}{\partial \theta} + \frac{1}{r^*} \mathbf{1}_{2i+1} \otimes \Lambda_2^* \frac{\partial \psi'}{\partial \theta} \\
& - \frac{i}{r^*} \Lambda_2^* \otimes \mathbf{1}_{2l+1} \frac{\partial \psi'}{\partial \tau} - \frac{i}{r^*} \mathbf{1}_{2i+1} \otimes \Lambda_2^* \frac{\partial \psi'}{\partial \tau} + \Lambda_3^* \frac{\partial \psi'}{\partial r^*} - i\Lambda_3^* \frac{\partial \psi'}{\partial r^*} \\
& + \left[\frac{1}{r^* \sin \theta^c} \Lambda_1^* \otimes \mathbf{1}_{2l+1} T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \varphi} - \frac{1}{r^* \sin \theta^c} \mathbf{1}_{2i+1} \otimes \Lambda_1^* T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \varphi} \right. \\
& + \frac{i}{r^* \sin \theta^c} \Lambda_1^* \otimes \mathbf{1}_{2l+1} T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \epsilon} + \frac{i}{r^* \sin \theta^c} \mathbf{1}_{2i+1} \otimes \Lambda_1^* T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \epsilon} \\
& - \frac{1}{r^*} \Lambda_2^* \otimes \mathbf{1}_{2l+1} T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \theta} + \frac{1}{r^*} \mathbf{1}_{2i+1} \otimes \Lambda_2^* T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \theta} \\
& \left. - \frac{i}{r^*} \Lambda_2^* \otimes \mathbf{1}_{2l+1} T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \tau} - \frac{i}{r^*} \mathbf{1}_{2i+1} \otimes \Lambda_2^* T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \tau} + \kappa^c I \right] \\
& \times \psi' = 0.
\end{aligned}$$

(98)

Let us show that the products $T_{ij}(\mathbf{g}) \frac{\partial T_{ij}^{-1}(\mathbf{g})}{\partial \varphi}$, \dots , $T_{ij}^*(\mathbf{g}) \frac{\partial T_{ij}^{*-1}(\mathbf{g})}{\partial \tau}$ are expressed via linear combinations of the infinitesimal operators. For example, let us consider the simplest tensor representation $\tau_{\frac{1}{2}, \frac{1}{2}} = \tau_{\frac{1}{2}, 0} \otimes \tau_{0, \frac{1}{2}}$. The representation $\tau_{\frac{1}{2}, \frac{1}{2}}$ is realized in the four-dimensional symmetric space $(1, 1)$. The matrix of $\tau_{\frac{1}{2}, \frac{1}{2}}$ in the space $(1, 1)$ has the following

form:

$$\begin{aligned} & \tau_{\frac{1}{2}\frac{1}{2}}^{-1}(\mathfrak{g}) \\ &= \begin{pmatrix} e^\epsilon \cos \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & -ie^{i\varphi} \cos \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & ie^{-i\varphi} \sin \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & e^{-\epsilon} \sin \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} \\ -ie^\epsilon \cos \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & e^{i\varphi} \cos \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & e^{-i\varphi} \sin \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & ie^{-\epsilon} \sin \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} \\ ie^\epsilon \sin \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & e^{i\varphi} \sin \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & e^{-i\varphi} \cos \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & -ie^{-\epsilon} \cos \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} \\ e^\epsilon \sin \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & ie^{i\varphi} \sin \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & -ie^{-i\varphi} \cos \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & e^{-\epsilon} \cos \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} \end{pmatrix}. \end{aligned}$$

This matrix is obtained from (28) via replacing all the functions $\mathfrak{Z}_{mn;\hat{m}\hat{n}}^{li}(\theta, \tau)$ by $\mathfrak{M}_{mn;\hat{m}\hat{n}}^{li}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{-m(\epsilon+i\varphi)-\hat{n}(\epsilon-i\varphi)} \mathfrak{Z}_{mn;\hat{m}\hat{n}}^{li}(\theta, \tau)$. An inverse matrix for $T_{\frac{1}{2}\frac{1}{2}}(\mathfrak{g})$ is

$$\begin{aligned} & \tau_{\frac{1}{2}\frac{1}{2}}^{-1}(\mathfrak{g}) \\ &= \begin{pmatrix} e^{-\epsilon} \cos \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & ie^{-\epsilon} \cos \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & -ie^{-\epsilon} \sin \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & e^{-\epsilon} \sin \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} \\ ie^{-i\varphi} \cos \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & e^{-i\varphi} \cos \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & e^{-i\varphi} \sin \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & -ie^{-i\varphi} \sin \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} \\ -ie^{i\varphi} \sin \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & e^{i\varphi} \sin \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & e^{i\varphi} \cos \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & ie^{i\varphi} \cos \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} \\ e^\epsilon \sin \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & -ie^\epsilon \sin \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} & ie^\epsilon \cos \frac{\theta^c}{2} \sin \frac{\hat{\theta}^c}{2} & e^\epsilon \cos \frac{\theta^c}{2} \cos \frac{\hat{\theta}^c}{2} \end{pmatrix}. \end{aligned}$$

Infinitesimal operators of the representations $\tau_{\frac{1}{2}, 0}$ and $\tau_{0, \frac{1}{2}}$ are

$$\begin{aligned} A_1^{\frac{1}{2}} &= -\frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & A_2^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & A_3^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \\ B_1^{\frac{1}{2}} &= -\frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & B_2^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, & B_3^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \tilde{A}_1^{\frac{1}{2}} &= -\frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \tilde{A}_2^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \tilde{A}_3^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \\ \tilde{B}_1^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \tilde{B}_2^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, & \tilde{B}_3^{\frac{1}{2}} &= \frac{1}{2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \quad (99)$$

Taking into account the latter relations, we find

$$\begin{aligned} & \tau_{\frac{1}{2}\frac{1}{2}}^{-1}(\mathfrak{g}) \frac{\partial \tau_{\frac{1}{2}\frac{1}{2}}^{-1}(\mathfrak{g})}{\partial \varphi} \\ &= \frac{1}{2} \begin{pmatrix} -i \cos \theta^c + i \cos \hat{\theta}^c & -\sin \hat{\theta}^c & -\sin \theta^c & 0 \\ \sin \theta^c & -i \cos \theta^c - i \cos \hat{\theta}^c & 0 & -\sin \theta^c \\ \sin \theta^c & 0 & i \cos \theta^c + i \cos \hat{\theta}^c & -\sin \hat{\theta}^c \\ 0 & \sin \theta^c & \sin \hat{\theta}^c & i \cos \theta^c - i \cos \hat{\theta}^c \end{pmatrix} \\ &= -(A_3 \otimes \mathbf{1}_2 \cos \theta^c - \mathbf{1}_2 \otimes \tilde{A}_3 \cos \hat{\theta}^c) - (A_2 \otimes \mathbf{1}_2 \sin \theta^c + \mathbf{1}_2 \otimes \tilde{A}_2 \sin \hat{\theta}^c), \end{aligned} \quad (100)$$

$$\begin{aligned}
& \tau_{\frac{1}{2}\frac{1}{2}}(\mathfrak{g}) \frac{\partial \tau_{\frac{1}{2}\frac{1}{2}}^{-1}(\mathfrak{g})}{\partial \epsilon} \\
&= \frac{1}{2} \begin{pmatrix} -\cos \theta^c - \cos \dot{\theta}^c & -i \sin \dot{\theta}^c & i \sin \theta^c & 0 \\ i \sin \dot{\theta}^c & -\cos \theta^c + \cos \dot{\theta}^c & 0 & i \sin \theta^c \\ -i \sin \theta^c & 0 & \cos \theta^c - \cos \dot{\theta}^c & -i \sin \dot{\theta}^c \\ 0 & -i \sin \theta^c & i \sin \dot{\theta}^c & \cos \theta^c + \cos \dot{\theta}^c \end{pmatrix} \\
&= -(\mathbf{B}_3 \otimes \mathbf{1}_2 \cos \theta^c - \mathbf{1}_2 \otimes \tilde{\mathbf{B}}_3 \cos \dot{\theta}^c) - (\mathbf{B}_2 \otimes \mathbf{1}_2 \sin \theta^c + \mathbf{1}_2 \otimes \tilde{\mathbf{B}}_2 \sin \dot{\theta}^c),
\end{aligned} \tag{101}$$

$$\tau_{\frac{1}{2}\frac{1}{2}}(\mathfrak{g}) \frac{\partial \tau_{\frac{1}{2}\frac{1}{2}}^{-1}(\mathfrak{g})}{\partial \theta} = \frac{1}{2} \begin{pmatrix} 0 & i & -i & 0 \\ i & 0 & 0 & -i \\ -i & 0 & 0 & i \\ 0 & -i & i & 0 \end{pmatrix} = \mathbf{A}_1 \otimes \mathbf{1}_2 - \mathbf{1}_2 \otimes \tilde{\mathbf{A}}_1, \tag{102}$$

$$\tau_{\frac{1}{2}\frac{1}{2}}(\mathfrak{g}) \frac{\partial \tau_{\frac{1}{2}\frac{1}{2}}^{-1}(\mathfrak{g})}{\partial \tau} = -\frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \mathbf{B}_1 \otimes \mathbf{1}_2 - \mathbf{1}_2 \otimes \tilde{\mathbf{B}}_1. \tag{103}$$

Further, a matrix of the conjugate representation $\tau_{\frac{1}{2}\frac{1}{2}}^*$ has the form

$$\begin{aligned}
& \tau_{\frac{1}{2}\frac{1}{2}}^*(\mathfrak{g}) \\
&= \begin{pmatrix} e^\epsilon \cos \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & i e^{-i\varphi} \cos \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & -i e^{i\varphi} \sin \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & e^{-\epsilon} \sin \frac{\theta^c}{2} \sin \frac{\theta^c}{2} \\ i e^\epsilon \cos \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^{-i\varphi} \cos \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & e^{i\varphi} \sin \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & -i e^{-\epsilon} \sin \frac{\theta^c}{2} \cos \frac{\theta^c}{2} \\ -i e^\epsilon \sin \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & e^{-i\varphi} \sin \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^{i\varphi} \cos \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & i e^{-\epsilon} \cos \frac{\theta^c}{2} \sin \frac{\theta^c}{2} \\ e^\epsilon \sin \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & -i e^{-i\varphi} \sin \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & i e^{i\varphi} \cos \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^{-\epsilon} \cos \frac{\theta^c}{2} \cos \frac{\theta^c}{2} \end{pmatrix},
\end{aligned}$$

and its inverse matrix is

$$\begin{aligned}
& \tau_{\frac{1}{2}\frac{1}{2}}^{*-1}(\mathfrak{g}) \\
&= \begin{pmatrix} e^{-\epsilon} \cos \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & -i e^{-\epsilon} \cos \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & i e^{-\epsilon} \sin \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & e^{-\epsilon} \sin \frac{\theta^c}{2} \sin \frac{\theta^c}{2} \\ -i e^{i\varphi} \cos \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^{i\varphi} \cos \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & e^{i\varphi} \sin \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & i e^{i\varphi} \sin \frac{\theta^c}{2} \cos \frac{\theta^c}{2} \\ i e^{-i\varphi} \sin \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & e^{-i\varphi} \sin \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^{-i\varphi} \cos \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & -i e^{-i\varphi} \cos \frac{\theta^c}{2} \sin \frac{\theta^c}{2} \\ e^\epsilon \sin \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & i e^\epsilon \sin \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & -i e^\epsilon \cos \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^\epsilon \cos \frac{\theta^c}{2} \cos \frac{\theta^c}{2} \end{pmatrix}.
\end{aligned}$$

In this case, we have

$$\begin{aligned}
 & \tau_{\frac{1}{2}\frac{1}{2}}^*(\mathbf{g}) \frac{\partial \tau_{\frac{1}{2}\frac{1}{2}}^{*-1}(\mathbf{g})}{\partial \varphi} \\
 &= \frac{1}{2} \begin{pmatrix} -i \cos \theta^c + i \cos \dot{\theta}^c & -\sin \theta^c & -\sin \dot{\theta}^c & 0 \\ \sin \theta^c & i \cos \theta^c + i \cos \dot{\theta}^c & 0 & -\sin \dot{\theta}^c \\ \sin \dot{\theta}^c & 0 & -i \cos \theta^c - i \cos \dot{\theta}^c & -\sin \theta^c \\ 0 & \sin \dot{\theta}^c & \sin \theta^c & i \cos \theta^c - i \cos \dot{\theta}^c \end{pmatrix} \\
 &= (\tilde{\mathbf{A}}_3 \otimes \mathbf{1}_2 \cos \dot{\theta}^c - \mathbf{1}_2 \otimes \mathbf{A}_3 \cos \theta^c) - (\tilde{\mathbf{A}}_2 \otimes \mathbf{1}_2 \sin \dot{\theta}^c + \mathbf{1}_2 \otimes \mathbf{A}_2 \sin \theta^c), \quad (104)
 \end{aligned}$$

$$\begin{aligned}
 & \tau_{\frac{1}{2}\frac{1}{2}}^*(\mathbf{g}) \frac{\partial \tau_{\frac{1}{2}\frac{1}{2}}^{*-1}(\mathbf{g})}{\partial \epsilon} \\
 &= \frac{1}{2} \begin{pmatrix} -\cos \theta^c - \cos \dot{\theta}^c & i \sin \theta^c & -i \sin \dot{\theta}^c & 0 \\ -i \sin \theta^c & \cos \theta^c - \cos \dot{\theta}^c & 0 & -i \sin \dot{\theta}^c \\ i \sin \dot{\theta}^c & 0 & -\cos \theta^c + \cos \dot{\theta}^c & i \sin \theta^c \\ 0 & i \sin \dot{\theta}^c & -i \sin \theta^c & \cos \theta^c + \cos \dot{\theta}^c \end{pmatrix} \\
 &= (\tilde{\mathbf{B}}_3 \otimes \mathbf{1}_2 \cos \dot{\theta}^c - \mathbf{1}_2 \otimes \mathbf{B}_3 \cos \theta^c) - (\tilde{\mathbf{B}}_2 \otimes \mathbf{1}_2 \sin \dot{\theta}^c + \mathbf{1}_2 \otimes \mathbf{B}_2 \sin \theta^c), \quad (105)
 \end{aligned}$$

$$\tau_{\frac{1}{2}\frac{1}{2}}^*(\mathbf{g}) \frac{\partial \tau_{\frac{1}{2}\frac{1}{2}}^{*-1}(\mathbf{g})}{\partial \theta} = \frac{1}{2} \begin{pmatrix} 0 & -i & i & 0 \\ -i & 0 & 0 & i \\ i & 0 & 0 & -i \\ 0 & i & -i & 0 \end{pmatrix} = -(\tilde{\mathbf{A}}_1 \otimes \mathbf{1}_2 - \mathbf{1}_2 \otimes \mathbf{A}_1), \quad (106)$$

$$\tau_{\frac{1}{2}\frac{1}{2}}^*(\mathbf{g}) \frac{\partial \tau_{\frac{1}{2}\frac{1}{2}}^{*-1}(\mathbf{g})}{\partial \tau} = -\frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = -(\tilde{\mathbf{B}}_1 \otimes \mathbf{1}_2 - \mathbf{1}_2 \otimes \mathbf{B}_1) \quad (107)$$

In the following example, we consider the first nontrivial tensor representation $\tau_{1, \frac{1}{2}} = \tau_{1,0} \otimes \tau_{0, \frac{1}{2}}$. The representation $\tau_{1, \frac{1}{2}}$ is realized in the six-dimensional symmetric space (2, 1). The matrix of $\tau_{1, \frac{1}{2}}$ in the space (2, 1) has the following form:

$$\tau_{1, \frac{1}{2}}(\mathbf{g}) = \begin{pmatrix} e^{\frac{3\epsilon+i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \cos \frac{\dot{\theta}^c}{2} & -ie^{\frac{\epsilon+3i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \sin \frac{\dot{\theta}^c}{2} & \frac{\epsilon-i\varphi}{\sqrt{2}} \sin \theta^c \cos \frac{\dot{\theta}^c}{2} \\ -ie^{\frac{3\epsilon+i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \sin \frac{\dot{\theta}^c}{2} & e^{\frac{\epsilon+3i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \cos \frac{\dot{\theta}^c}{2} & \frac{\epsilon-i\varphi}{\sqrt{2}} \sin \theta^c \sin \frac{\dot{\theta}^c}{2} \\ \frac{\epsilon+i\varphi}{\sqrt{2}} \sin \theta^c \cos \frac{\dot{\theta}^c}{2} & \frac{\epsilon+3i\varphi}{\sqrt{2}} \sin \theta^c \sin \frac{\dot{\theta}^c}{2} & e^{\frac{\epsilon-i\varphi}{2}} \cos \theta^c \cos \frac{\dot{\theta}^c}{2} \\ \frac{\epsilon+3i\varphi}{\sqrt{2}} \sin \theta^c \sin \frac{\dot{\theta}^c}{2} & \frac{\epsilon+3i\varphi}{\sqrt{2}} \sin \theta^c \cos \frac{\dot{\theta}^c}{2} & -ie^{\frac{\epsilon-i\varphi}{2}} \cos \theta^c \sin \frac{\dot{\theta}^c}{2} \\ -e^{\frac{3\epsilon+i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \cos \frac{\dot{\theta}^c}{2} & ie^{\frac{\epsilon+3i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \sin \frac{\dot{\theta}^c}{2} & \frac{\epsilon-i\varphi}{\sqrt{2}} \sin \theta^c \cos \frac{\dot{\theta}^c}{2} \\ ie^{\frac{3\epsilon+i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \sin \frac{\dot{\theta}^c}{2} & -e^{\frac{\epsilon+3i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \cos \frac{\dot{\theta}^c}{2} & \frac{\epsilon-i\varphi}{\sqrt{2}} \sin \theta^c \sin \frac{\dot{\theta}^c}{2} \end{pmatrix}$$

$$\left(\begin{array}{ccc} e^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} & -e^{-\frac{\epsilon+3i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & ie^{-\frac{3\epsilon+i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} \\ ie^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} & ie^{-\frac{\epsilon+3i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & -e^{-\frac{3\epsilon+i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} \\ -ie^{-\frac{-\epsilon+i\varphi}{2}} \cos \theta^c \sin \frac{\theta^c}{2} & ie^{-\frac{\epsilon+3i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} & e^{-\frac{3\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} \\ e^{-\frac{-\epsilon+i\varphi}{2}} \cos \theta^c \cos \frac{\theta^c}{2} & e^{-\frac{\epsilon+3i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} & ie^{-\frac{3\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} \\ e^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} & e^{-\frac{\epsilon+3i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & -ie^{-\frac{3\epsilon+i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} \\ ie^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} & -ie^{-\frac{\epsilon+3i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^{-\frac{3\epsilon+i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} \end{array} \right)$$

In turn, this matrix is obtained from (29) via replacing all the functions $\mathfrak{J}_{mn;mn}^{lj}(\theta, \tau)$ by $\mathfrak{M}_{mn;mn}^{lj}(\varphi, \epsilon, \theta, \tau, 0, 0)$. An inverse matrix for $T_{1\frac{1}{2}}(\mathfrak{g})$ is

$$\tau_{1\frac{1}{2}}^{-1}(\mathfrak{g}) = \left(\begin{array}{ccc} e^{-\frac{3\epsilon+i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & ie^{-\frac{3\epsilon+i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & -ie^{-\frac{3\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} \\ ie^{-\frac{\epsilon+3i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^{-\frac{\epsilon+3i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & e^{-\frac{\epsilon+3i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} \\ -ie^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} & e^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} & e^{-\frac{-\epsilon+i\varphi}{2}} \cos \theta^c \cos \frac{\theta^c}{2} \\ e^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} & -ie^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} & ie^{-\frac{-\epsilon+i\varphi}{2}} \cos \theta^c \sin \frac{\theta^c}{2} \\ -e^{-\frac{\epsilon+3i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & -ie^{-\frac{\epsilon+3i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & -ie^{-\frac{\epsilon+3i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} \\ -ie^{-\frac{3\epsilon+i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & -e^{-\frac{3\epsilon+i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & e^{-\frac{3\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} \\ e^{-\frac{3\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} & -e^{-\frac{3\epsilon+i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & -ie^{-\frac{3\epsilon+i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} \\ -ie^{-\frac{\epsilon+3i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} & -ie^{-\frac{\epsilon+3i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & -e^{-\frac{\epsilon+3i\varphi}{2}} \sin^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} \\ ie^{-\frac{-\epsilon+i\varphi}{2}} \cos \theta^c \sin \frac{\theta^c}{2} & -ie^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} & e^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} \\ e^{-\frac{-\epsilon+i\varphi}{2}} \cos \theta^c \cos \frac{\theta^c}{2} & e^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} & -ie^{-\frac{-\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} \\ e^{-\frac{\epsilon+3i\varphi}{2}} \sin \theta^c \sin \frac{\theta^c}{2} & e^{-\frac{\epsilon+3i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} & ie^{-\frac{\epsilon+3i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} \\ -ie^{-\frac{3\epsilon+i\varphi}{2}} \sin \theta^c \cos \frac{\theta^c}{2} & ie^{-\frac{3\epsilon+i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \sin \frac{\theta^c}{2} & e^{-\frac{3\epsilon+i\varphi}{2}} \cos^2 \frac{\theta^c}{2} \cos \frac{\theta^c}{2} \end{array} \right)$$

In turn, infinitesimal operators of the representation $\tau_{1,0}$ are

$$A_1^1 = -\frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad A_3^1 = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix},$$

$$B_1^1 = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad B_3^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\tilde{A}_1^1 = -\frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tilde{A}_2^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \tilde{A}_3^1 = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix},$$

$$\tilde{B}_1^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tilde{B}_2^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{bmatrix}, \quad \tilde{B}_3^1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Taking into account the latter expressions and the operators in (99), we obtain

$$\tau_{1\frac{1}{2}}(\mathbf{g}) \frac{\partial \tau_{1\frac{1}{2}}^{-1}(\mathbf{g})}{\partial \varphi} = \begin{pmatrix} -i \cos \theta^c + \frac{i}{2} \cos \theta^c & -\frac{1}{2} \sin \theta^c & -\frac{1}{\sqrt{2}} \sin \theta^c & 0 & 0 & 0 \\ \frac{1}{2} \sin \theta^c & -i \cos \theta^c - \frac{i}{2} \cos \theta^c & 0 & -\frac{1}{\sqrt{2}} \sin \theta^c & 0 & 0 \\ \frac{1}{\sqrt{2}} \sin \theta^c & 0 & \frac{i}{2} \cos \theta^c & -\frac{1}{2} \sin \theta^c & -\frac{1}{\sqrt{2}} \sin \theta^c & 0 \\ 0 & \frac{1}{\sqrt{2}} \sin \theta^c & \frac{1}{2} \sin \theta^c & -\frac{i}{2} \cos \theta^c & 0 & -\frac{1}{\sqrt{2}} \sin \theta^c \\ 0 & 0 & \frac{1}{\sqrt{2}} \sin \theta^c & 0 & i \cos \theta^c + \frac{i}{2} \cos \theta^c & -\frac{1}{2} \sin \theta^c \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \sin \theta^c & \frac{1}{2} \sin \theta^c & i \cos \theta^c - \frac{i}{2} \cos \theta^c \end{pmatrix}$$

$$= -\left(\mathbf{A}_3^1 \otimes \mathbf{1}_2 \cos \theta^c - \mathbf{1}_3 \otimes \tilde{\mathbf{A}}_3^{\frac{1}{2}} \cos \theta^c \right) - \left(\mathbf{A}_2^1 \otimes \mathbf{1}_2 \sin \theta^c + \mathbf{1}_3 \otimes \tilde{\mathbf{A}}_2^{\frac{1}{2}} \sin \theta^c \right), \tag{108}$$

$$\tau_{1\frac{1}{2}}(\mathbf{g}) \frac{\partial \tau_{1\frac{1}{2}}^{-1}(\mathbf{g})}{\partial \epsilon} = \begin{pmatrix} -\cos \theta^c - \frac{1}{2} \cos \theta^c & -\frac{i}{2} \sin \theta^c & -\frac{i}{\sqrt{2}} \sin \theta^c & 0 & 0 & 0 \\ \frac{i}{2} \sin \theta^c & -\cos \theta^c + \frac{1}{2} \cos \theta^c & 0 & \frac{i}{\sqrt{2}} \sin \theta^c & 0 & 0 \\ -\frac{i}{\sqrt{2}} \sin \theta^c & 0 & -\frac{1}{2} \cos \theta^c & -\frac{i}{2} \sin \theta^c & \frac{i}{\sqrt{2}} \sin \theta^c & 0 \\ 0 & -\frac{i}{\sqrt{2}} \sin \theta^c & \frac{i}{2} \sin \theta^c & \frac{1}{2} \cos \theta^c & 0 & \frac{i}{\sqrt{2}} \sin \theta^c \\ 0 & 0 & -\frac{i}{\sqrt{2}} \sin \theta^c & 0 & \cos \theta^c - \frac{1}{2} \cos \theta^c & -\frac{i}{2} \sin \theta^c \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} \sin \theta^c & \frac{i}{2} \sin \theta^c & \cos \theta^c + \frac{1}{2} \cos \theta^c \end{pmatrix}$$

$$= -\left(\mathbf{B}_3^1 \otimes \mathbf{1}_2 \cos \theta^c - \mathbf{1}_3 \otimes \tilde{\mathbf{B}}_3^{\frac{1}{2}} \cos \theta^c \right) - \left(\mathbf{B}_2^1 \otimes \mathbf{1}_2 \sin \theta^c + \mathbf{1}_3 \otimes \tilde{\mathbf{B}}_2^{\frac{1}{2}} \sin \theta^c \right), \tag{109}$$

$$\tau_{1\frac{1}{2}}(\mathbf{g}) \frac{\partial \tau_{1\frac{1}{2}}^{-1}(\mathbf{g})}{\partial \theta} = \begin{pmatrix} 0 & \frac{i}{2} & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{2} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & -\frac{i}{\sqrt{2}} & \frac{i}{2} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & \frac{i}{2} & 0 \end{pmatrix}$$

$$= \mathbf{A}_1^1 \otimes \mathbf{1}_2 - \mathbf{1}_3 \otimes \tilde{\mathbf{A}}_1^{\frac{1}{2}}, \tag{110}$$

$$\begin{aligned} \tau_{1\frac{1}{2}}(\mathfrak{g}) \frac{\partial \tau_{1\frac{1}{2}}^{-1}(\mathfrak{g})}{\partial \tau} &= \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \end{pmatrix} \\ &= \mathbf{B}_1^1 \otimes \mathbf{1}_2 - \mathbf{1}_3 \otimes \tilde{\mathbf{B}}_1^{\frac{1}{2}}. \end{aligned} \tag{111}$$

Further, the conjugate representation $\tau_{1,\frac{1}{2}}^*$ acts in the space (1, 2) and we have the following relations:

$$\begin{aligned} \tau_{1\frac{1}{2}}^*(\mathfrak{g}) \frac{\partial \tau_{1\frac{1}{2}}^{*-1}(\mathfrak{g})}{\partial \varphi} &= \begin{pmatrix} i \cos \theta^c - \frac{1}{2} \cos \theta^c & -\frac{1}{2} \sin \theta^c & -\frac{1}{\sqrt{2}} \sin \theta^c & 0 & 0 & 0 \\ \frac{1}{2} \sin \theta^c & i \cos \theta^c + \frac{1}{2} \cos \theta^c & 0 & -\frac{1}{\sqrt{2}} \sin \theta^c & 0 & 0 \\ -\frac{1}{\sqrt{2}} \sin \theta^c & 0 & -\frac{i}{2} \cos \theta^c & -\frac{1}{2} \sin \theta^c & -\frac{1}{\sqrt{2}} \sin \theta^c & 0 \\ 0 & \frac{1}{\sqrt{2}} \sin \theta^c & \frac{1}{2} \sin \theta^c & \frac{1}{2} \cos \theta^c & 0 & -\frac{1}{\sqrt{2}} \sin \theta^c \\ 0 & 0 & \frac{1}{\sqrt{2}} \sin \theta^c & 0 & -i \cos \theta^c - \frac{i}{2} \cos \theta^c & -\frac{1}{2} \sin \theta^c \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \sin \theta^c & \frac{1}{2} \sin \theta^c & -i \cos \theta^c + \frac{i}{2} \cos \theta^c \end{pmatrix} \\ &= \left(\tilde{\mathbf{A}}_3^1 \otimes \mathbf{1}_2 \cos \theta^c - \mathbf{1}_3 \otimes \mathbf{A}_3^{\frac{1}{2}} \cos \theta^c \right) - \left(\tilde{\mathbf{A}}_2^1 \otimes \mathbf{1}_2 \sin \theta^c + \mathbf{1}_3 \otimes \mathbf{A}_2^{\frac{1}{2}} \sin \theta^c \right), \end{aligned} \tag{112}$$

$$\begin{aligned} \tau_{1\frac{1}{2}}^*(\mathfrak{g}) \frac{\partial \tau_{1\frac{1}{2}}^{*-1}(\mathfrak{g})}{\partial \epsilon} &= \begin{pmatrix} -\cos \theta^c - \frac{1}{2} \cos \theta^c & \frac{i}{2} \sin \theta^c & -\frac{i}{\sqrt{2}} \sin \theta^c & 0 & 0 & 0 \\ -\frac{i}{2} \sin \theta^c & -\cos \theta^c + \frac{1}{2} \cos \theta^c & 0 & -\frac{i}{\sqrt{2}} \sin \theta^c & 0 & 0 \\ \frac{i}{\sqrt{2}} \sin \theta^c & 0 & -\frac{1}{2} \cos \theta^c & \frac{i}{2} \sin \theta^c & -\frac{i}{\sqrt{2}} \sin \theta^c & 0 \\ 0 & \frac{i}{\sqrt{2}} \sin \theta^c & -\frac{i}{2} \sin \theta^c & \frac{1}{2} \cos \theta^c & 0 & -\frac{i}{\sqrt{2}} \sin \theta^c \\ 0 & 0 & \frac{i}{\sqrt{2}} \sin \theta^c & 0 & \cos \theta^c - \frac{1}{2} \cos \theta^c & \frac{i}{2} \sin \theta^c \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} \sin \theta^c & -\frac{i}{2} \sin \theta^c & \cos \theta^c + \frac{1}{2} \cos \theta^c \end{pmatrix} \\ &= \left(\tilde{\mathbf{B}}_3^1 \otimes \mathbf{1}_2 \cos \theta^c - \mathbf{1}_3 \otimes \mathbf{B}_3^{\frac{1}{2}} \cos \theta^c \right) - \left(\tilde{\mathbf{B}}_2^1 \otimes \mathbf{1}_2 \sin \theta^c + \mathbf{1}_3 \otimes \mathbf{B}_2^{\frac{1}{2}} \sin \theta^c \right), \end{aligned} \tag{113}$$

$$\begin{aligned} \tau_{1\frac{1}{2}}^*(\mathfrak{g}) \frac{\partial \tau_{1\frac{1}{2}}^{*-1}(\mathfrak{g})}{\partial \theta} &= \begin{pmatrix} 0 & -\frac{i}{2} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 & -\frac{i}{2} & \frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & -\frac{i}{2} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & -\frac{i}{2} & 0 \end{pmatrix} \\ &= -\left(\tilde{\mathbf{A}}_1^1 \otimes \mathbf{1}_2 - \mathbf{1}_3 \otimes \mathbf{A}_1^{\frac{1}{2}} \right), \end{aligned} \tag{114}$$

$$\begin{aligned}
 \tau_{1\frac{1}{2}}^*(\mathfrak{g}) \frac{\partial \tau_{1\frac{1}{2}}^{*-1}(\mathfrak{g})}{\partial \tau} &= \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \end{pmatrix} \\
 &= -\left(\tilde{\mathbf{B}}_1^1 \otimes \mathbf{1}_2 - \mathbf{1}_3 \otimes \mathbf{B}_1^{\frac{1}{2}} \right)
 \end{aligned} \tag{115}$$

It is easy to verify that relations of the type (100)–(115) take place for any representation $T_{ii}(\mathfrak{g})$ of the group \mathfrak{G}_+ . Therefore,

$$\begin{aligned}
 \tau_{ii}(\mathfrak{g}) \frac{\partial T_{ii}^{-1}(\mathfrak{g})}{\partial \varphi} &= -\left(\mathbf{A}_3^l \otimes \mathbf{1}_{2l+1} \cos \theta^c - \mathbf{1}_{2l+1} \otimes \mathbf{A}_3^i \cos \theta^c \right) \\
 &\quad - \left(\mathbf{A}_2^l \otimes \mathbf{1}_{2l+1} \sin \theta^c + \mathbf{1}_{2l+1} \otimes \mathbf{A}_2^i \sin \theta^c \right), \\
 \tau_{ii}(\mathfrak{g}) \frac{\partial T_{ii}^{-1}(\mathfrak{g})}{\partial \epsilon} &= -\left(\mathbf{B}_3^l \otimes \mathbf{1}_{2l+1} \cos \theta^c - \mathbf{1}_{2l+1} \otimes \mathbf{B}_3^i \cos \theta^c \right) \\
 &\quad - \left(\mathbf{B}_2^l \otimes \mathbf{1}_{2l+1} \sin \theta^c + \mathbf{1}_{2l+1} \otimes \mathbf{B}_2^i \sin \theta^c \right), \\
 \tau_{ii}(\mathfrak{g}) \frac{\partial T_{ii}^{-1}(\mathfrak{g})}{\partial \theta} &= \mathbf{A}_1^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \mathbf{A}_1^i, \\
 \tau_{ii}(\mathfrak{g}) \frac{\partial T_{ii}^{-1}(\mathfrak{g})}{\partial \tau} &= \mathbf{B}_1^l \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \mathbf{B}_1^i, \\
 \tau_{ii}^*(\mathfrak{g}) \frac{\partial \tau_{ii}^{*-1}(\mathfrak{g})}{\partial \varphi} &= \left(\mathbf{A}_3^i \otimes \mathbf{1}_{2l+1} \cos \theta^c - \mathbf{1}_{2l+1} \otimes \mathbf{A}_3^l \cos \theta^c \right) \\
 &\quad - \left(\mathbf{A}_2^i \otimes \mathbf{1}_{2l+1} \sin \theta^c + \mathbf{1}_{2l+1} \otimes \mathbf{A}_2^l \sin \theta^c \right), \\
 \tau_{ii}^*(\mathfrak{g}) \frac{\partial \tau_{ii}^{*-1}(\mathfrak{g})}{\partial \epsilon} &= \left(\mathbf{B}_3^i \otimes \mathbf{1}_{2l+1} \cos \theta^c - \mathbf{1}_{2l+1} \otimes \mathbf{B}_3^l \cos \theta^c \right) \\
 &\quad - \left(\mathbf{B}_2^i \otimes \mathbf{1}_{2l+1} \sin \theta^c + \mathbf{1}_{2l+1} \otimes \mathbf{B}_2^l \sin \theta^c \right), \\
 \tau_{ii}^*(\mathfrak{g}) \frac{\partial \tau_{ii}^{*-1}(\mathfrak{g})}{\partial \theta} &= -\left(\mathbf{A}_1^i \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \mathbf{A}_1^l \right), \\
 \tau_{ii}^*(\mathfrak{g}) \frac{\partial \tau_{ii}^{*-1}(\mathfrak{g})}{\partial \tau} &= -\left(\mathbf{B}_1^i \otimes \mathbf{1}_{2l+1} - \mathbf{1}_{2l+1} \otimes \mathbf{B}_1^l \right)
 \end{aligned}$$

Substituting these relations into system (98), we obtain

$$\begin{aligned}
& \frac{1}{r \sin \theta^c} \Lambda_1^l \otimes \mathbf{1}_{2j+1} \frac{\partial \psi'}{\partial \varphi} - \frac{1}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes \Lambda_1^l \frac{\partial \psi'}{\partial \varphi} + \frac{i}{r \sin \theta^c} \Lambda_1^l \otimes \mathbf{1}_{2j+1} \frac{\partial \psi'}{\partial \epsilon} \\
& + \frac{i}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes \Lambda_1^l \frac{\partial \psi'}{\partial \epsilon} - \frac{1}{r} \Lambda_2^l \otimes \mathbf{1}_{2j+1} \frac{\partial \psi'}{\partial \theta} + \frac{1}{r} \mathbf{1}_{2l+1} \otimes \Lambda_2^l \frac{\partial \psi'}{\partial \theta} \\
& - \frac{i}{r} \Lambda_2^l \otimes \mathbf{1}_{2j+1} \frac{\partial \psi'}{\partial \tau} - \frac{i}{r} \mathbf{1}_{2l+1} \otimes \Lambda_2^l \frac{\partial \psi'}{\partial \tau} + (1+i) \Lambda_3^l \frac{\partial \psi'}{\partial r} \\
& + \frac{1}{r} \left[\Lambda_1^{ii} \Lambda_2^{ij} - \Lambda_2^{ii} \Lambda_1^{ij} + \Lambda_1^{ij} \mathbf{B}_2^{ij} - \Lambda_2^{ij} \mathbf{B}_1^{ij} - 2 \cot \theta^c \Lambda_1^l \otimes \mathbf{1}_{2l+1} \Lambda_3^l \otimes \mathbf{1}_{2j+1} \right. \\
& \left. - 2 \cot \theta^c \mathbf{1}_{2l+1} \otimes \Lambda_1^l \mathbf{1}_{2j+1} \otimes \Lambda_3^l \right] \psi' + \kappa^c \psi = 0, \\
& \frac{1}{r^* \sin \theta^c} \Lambda_1^{*i} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi'}{\partial \varphi} - \frac{1}{r^* \sin \theta^c} \mathbf{1}_{2i+1} \otimes \Lambda_1^{*i} \frac{\partial \psi'}{\partial \varphi} + \frac{i}{r^* \sin \theta^c} \Lambda_1^{*i} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi'}{\partial \epsilon} \\
& + \frac{i}{r^* \sin \theta^c} \mathbf{1}_{2i+1} \otimes \Lambda_1^{*i} \frac{\partial \psi'}{\partial \epsilon} - \frac{1}{r^*} \Lambda_2^{*i} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi'}{\partial \theta} + \frac{1}{r^*} \mathbf{1}_{2i+1} \otimes \Lambda_2^{*i} \frac{\partial \psi'}{\partial \theta} \\
& - \frac{i}{r^*} \Lambda_2^{*i} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi'}{\partial \tau} - \frac{i}{r^*} \mathbf{1}_{2i+1} \otimes \Lambda_2^{*i} \frac{\partial \psi'}{\partial \tau} + (1-i) \Lambda_3^{*i} \frac{\partial \psi'}{\partial r^*} \\
& + \frac{1}{r^*} \left[\Lambda_1^{*ii} \tilde{\Lambda}_2^{ij} - \Lambda_2^{*ii} \tilde{\Lambda}_1^{ij} + \Lambda_1^{*ij} \tilde{\mathbf{B}}_2^{ij} - \Lambda_2^{*ij} \tilde{\mathbf{B}}_1^{ij} - 2 \cot \theta^c \Lambda_1^{*i} \otimes \mathbf{1}_{2l+1} \Lambda_3^j \otimes \mathbf{1}_{2j+1} \right. \\
& \left. - 2 \cot \theta^c \mathbf{1}_{2i+1} \otimes \Lambda_1^{*i} \mathbf{1}_{2j+1} \otimes \Lambda_3^j \right] \psi' + \kappa^c \psi = 0. \tag{116}
\end{aligned}$$

Now, we can separate the variables in the relativistically invariant system. Namely, we represent the each component $\psi_{lm;jm}^{kk}$ of the wave function ψ in the form of an expansion in the generalized hyperspherical functions $\mathfrak{M}_{mn;jm}^{ii}$. This procedure gives rise to separation of variables, that is, it reduces the relativistically invariant system to the system of ordinary differential equations. Preliminarily, we will calculate elements of the matrices $D = \Lambda_1^{ii} \Lambda_2^{ij} - \Lambda_2^{ii} \Lambda_1^{ij}$, $E = \Lambda_1^{ij} \mathbf{B}_2^{ij} - \Lambda_2^{ij} \mathbf{B}_1^{ij}$, $D^* = \Lambda_1^{*ii} \tilde{\Lambda}_2^{ij} - \Lambda_2^{*ii} \tilde{\Lambda}_1^{ij}$, $E^* = \Lambda_1^{*ij} \tilde{\mathbf{B}}_2^{ij} - \Lambda_2^{*ij} \tilde{\mathbf{B}}_1^{ij}$. First of all, let us find elements of the matrix $D = \Lambda_1^{ii} \Lambda_2^{ij} - \Lambda_2^{ii} \Lambda_1^{ij}$. Using the relations (62), we can write $D = 2\Lambda_3^{ii} + \Lambda_2^{ij} \Lambda_1^{ij} - \Lambda_1^{ii} \Lambda_2^{ij}$. As usual, the action of the transformation D in the helicity basis has the following form:

$$D \zeta_{lm;jm}^{kk} = \sum_{l',m',i',\dot{m}',k',k'} a_{l',m',i',\dot{m}',k',k'}^{k'k;k'k} \zeta_{l',m',i',\dot{m}'}^{k'k'}$$

Taking into account (72), (74), (75), (8) and (9), we obtain

$$\begin{aligned}
D \zeta_{lm;jm}^{kk} &= \left(2\Lambda_3^{ii} + \Lambda_2^{ij} \Lambda_1^{ij} - \Lambda_1^{ii} \Lambda_2^{ij} \right) \zeta_{lm;jm}^{kk} \\
&= 2 \sum_{l',m',i',\dot{m}',k',k'} c_{l',m',i',\dot{m}',k',k'}^{k'k;k'k} \zeta_{l',m',i',\dot{m}'}^{k'k'} + \Lambda_2^{ij} \sum_{l',m',i',\dot{m}',k',k'} a_{l',m',i',\dot{m}',k',k'}^{k'k;k'k} \zeta_{l',m',i',\dot{m}'}^{k'k'}
\end{aligned}$$

$$\begin{aligned}
 & - A_1^i \sum_{l',m',i',\dot{m}',k',\dot{k}'} b_{l',m',m';i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \zeta_{l',m';i'\dot{m}'}^{k'\dot{k}'} \\
 & = 2 \sum_{l',m',i',\dot{m}',k',\dot{k}'} c_{l',m',m';i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \zeta_{l',m';i'\dot{m}'}^{k'\dot{k}'} + \frac{1}{2} \sum_{l',m',i',\dot{m}',k',\dot{k}'} a_{l',m',m';i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \left(\alpha_{m'}^{l'} \zeta_{l',m'-1;i'\dot{m}'}^{k'\dot{k}'} \right. \\
 & - \alpha_{m'+1}^{l'} \zeta_{l',m'+1;i'\dot{m}'}^{k'\dot{k}'} - \alpha_{\dot{m}'}^i \zeta_{l',m';i',\dot{m}'-1}^{k'\dot{k}'} + \alpha_{\dot{m}'+1}^i \zeta_{l',m';i',\dot{m}'+1}^{k'\dot{k}'} \left. \right) \\
 & - \frac{i}{2} \sum_{l',m',i',\dot{m}',k',\dot{k}'} b_{l',m',m';i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \left(-\alpha_{m'}^{l'} \zeta_{l',m'-1;i'\dot{m}'}^{k'\dot{k}'} - \alpha_{m'+1}^{l'} \zeta_{l',m'+1;i'\dot{m}'}^{k'\dot{k}'} \right. \\
 & \left. + \alpha_{\dot{m}'}^i \zeta_{l',m';i',\dot{m}'-1}^{k'\dot{k}'} + \alpha_{\dot{m}'+1}^i \zeta_{l',m';i',\dot{m}'+1}^{k'\dot{k}'} \right)
 \end{aligned}$$

Dividing each of the two latter sums on the four and changing the summation index in the each eight sums obtained, we come to the following expression:

$$\begin{aligned}
 D \zeta_{lm;\dot{m}}^{kk} = & \sum_{l',m',i',\dot{m}',k',\dot{k}'} \left(2c_{l',m',m';i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} + \frac{1}{2} \alpha_{m'+1}^{l'} a_{l',m'+1,m;i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \right. \\
 & - \frac{1}{2} \alpha_{m'}^{l'} a_{l',m'-1,m;i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} - \frac{1}{2} \alpha_{\dot{m}'+1}^i a_{l',m',m';\dot{m}'+1,\dot{m}}^{k'k;k'\dot{k}} \\
 & + \frac{1}{2} \alpha_{\dot{m}'}^i a_{l',m',m';i',\dot{m}'-1,\dot{m}}^{k'k;k'\dot{k}} + \frac{i}{2} \alpha_{m'+1}^{l'} b_{l',m'+1,m;i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \\
 & + \frac{i}{2} \alpha_{m'}^{l'} b_{l',m'-1,m;i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} - \frac{i}{2} \alpha_{\dot{m}'+1}^i b_{l',m',m';i',\dot{m}'+1,\dot{m}}^{k'k;k'\dot{k}} \\
 & \left. - \frac{i}{2} \alpha_{\dot{m}'}^i b_{l',m',m';i',\dot{m}'-1,\dot{m}}^{k'k;k'\dot{k}} \right) \zeta_{l',m';i'\dot{m}'}^{k'\dot{k}'}
 \end{aligned}$$

Therefore, a general element of the matrix D has the form

$$\begin{aligned}
 a_{l',m',m';i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} = & 2c_{l',m',m';i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} + \frac{1}{2} \alpha_{m'+1}^{l'} a_{l',m'+1,m;i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \\
 & - \frac{1}{2} \alpha_{m'}^{l'} a_{l',m'-1,m;i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} - \frac{1}{2} \alpha_{\dot{m}'+1}^i a_{l',m',m';\dot{m}'+1,\dot{m}}^{k'k;k'\dot{k}} \\
 & + \frac{1}{2} \alpha_{\dot{m}'}^i a_{l',m',m';i',\dot{m}'-1,\dot{m}}^{k'k;k'\dot{k}} + \frac{i}{2} \alpha_{m'+1}^{l'} b_{l',m'+1,m;i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} \\
 & + \frac{i}{2} \alpha_{m'}^{l'} b_{l',m'-1,m;i',\dot{m}'\dot{m}}^{k'k;k'\dot{k}} - \frac{i}{2} \alpha_{\dot{m}'+1}^i b_{l',m',m';i',\dot{m}'+1,\dot{m}}^{k'k;k'\dot{k}} \\
 & - \frac{i}{2} \alpha_{\dot{m}'}^i b_{l',m',m';i',\dot{m}'-1,\dot{m}}^{k'k;k'\dot{k}}
 \end{aligned}$$

Using the formulae (72), (74) and (75), we find that

$$D : \begin{cases} d_{l-1,l,mm;j-1,i,\dot{m}\dot{m}}^{k'k;k'\dot{k}} = c_{l-1,l;j-1,i}^{k'k;k'\dot{k}}(l+i)\sqrt{(l^2-m^2)(l^2-\dot{m}^2)}, \\ d_{l+1,l,mm;j+1,i,\dot{m}\dot{m}}^{k'k;k'\dot{k}} = -c_{l+1,l;j+1,i}^{k'k;k'\dot{k}}(l+i+2)\sqrt{((l+1)^2-m^2)((l+1)^2-\dot{m}^2)}. \end{cases} \quad (117)$$

All other elements $d_{l',m';j,i,\dot{m}'\dot{m}}^{k'k;k'\dot{k}}$ are equal to zero.

Analogously, using the relations (63), (66) and (67) and the operators (8)–(19), we find that elements of the matrices $E = -2i\Lambda_3^l + B_2^l\Lambda_1^l - B_1^l\Lambda_2^l$, $D = {}^*\Lambda_3^l + \tilde{A}_2^l\Lambda_1^l - \tilde{A}_1^l\Lambda_2^l$, $E = 2i\Lambda_3 + \tilde{B}_2^l\Lambda_1^l - \tilde{B}_1^l\Lambda_2^l$ are

$$E : \begin{cases} e_{l-1,l,mm;j-1,i,\dot{m}\dot{m}}^{k'k;k'\dot{k}} = ic_{l-1,l;j-1,i}^{k'k;k'\dot{k}}(l-l-2)\sqrt{(l^2-m^2)(l^2-\dot{m}^2)}, \\ e_{l,mm;jl,\dot{m}\dot{m}}^{k'k;k'\dot{k}} = -2ic_{ll;jl}^{k'k;k'\dot{k}}m\dot{m}, \\ e_{l+1,l,mm;j+1,i,\dot{m}\dot{m}}^{k'k;k'\dot{k}} = ic_{l+1,l;j+1,i}^{k'k;k'\dot{k}}(l-l-2)\sqrt{((l+1)^2-m^2)((l+1)^2-\dot{m}^2)}. \end{cases} \quad (118)$$

$${}^*D : \begin{cases} d_{l-1,l,mm;j-1,i,\dot{m}\dot{m}}^{*k'k;k'\dot{k}} = c_{l-1,l;j-1,i}^{*k'k;k'\dot{k}}(l+i)\sqrt{(l^2-m^2)(l^2-\dot{m}^2)}, \\ d_{l+1,l,mm;j+1,i,\dot{m}\dot{m}}^{*k'k;k'\dot{k}} = -c_{l+1,l;j+1,i}^{*k'k;k'\dot{k}}(l+i+2)\sqrt{((l+1)^2-m^2)((l+1)^2-\dot{m}^2)}. \end{cases} \quad (119)$$

$${}^*E : \begin{cases} e_{l-1,l,mm;j-1,i,\dot{m}\dot{m}}^{*k'k;k'\dot{k}} = ic_{l-1,l;j-1,i}^{*k'k;k'\dot{k}}(l-l+2)\sqrt{(l^2-m^2)(l^2-\dot{m}^2)}, \\ e_{l,mm;jl,\dot{m}\dot{m}}^{*k'k;k'\dot{k}} = 2ic_{ll;jl}^{*k'k;k'\dot{k}}m\dot{m}, \\ e_{l+1,l,mm;j+1,i,\dot{m}\dot{m}}^{*k'k;k'\dot{k}} = ic_{l+1,l;j+1,i}^{*k'k;k'\dot{k}}(l-l+2)\sqrt{((l+1)^2-m^2)((l+1)^2-\dot{m}^2)}. \end{cases} \quad (120)$$

The system (116) in the components $\psi_{lm;\dot{l}\dot{m}}^{k\dot{k}}$ can be written as

$$\begin{aligned} & \sum_{l',m';\dot{l}',\dot{m}',k',\dot{k}'} \left[\frac{1}{r \sin \theta^c} a_{ll',mm'}^{kk'} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi_{l'm';\dot{l}'\dot{m}'}^{k'k'}}{\partial \varphi} - \frac{1}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes a_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{k\dot{k}} \frac{\partial \psi_{l'm';\dot{l}'\dot{m}'}^{k'k'}}{\partial \varphi} \right. \\ & + \frac{i}{r \sin \theta^c} a_{ll',mm'}^{kk'} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi_{l'm';\dot{l}'\dot{m}'}^{k'k'}}{\partial \epsilon} + \frac{i}{r \sin \theta^c} \mathbf{1}_{2l+1} \otimes a_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{k\dot{k}} \frac{\partial \psi_{l'm';\dot{l}'\dot{m}'}^{k'k'}}{\partial \epsilon} \\ & - \frac{1}{r} b_{ll',mm'}^{kk'} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi_{l'm';\dot{l}'\dot{m}'}^{k'k'}}{\partial \theta} + \frac{1}{r} \mathbf{1}_{2l+1} \otimes b_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{k\dot{k}} \frac{\partial \psi_{l'm';\dot{l}'\dot{m}'}^{k'k'}}{\partial \theta} \\ & \left. - \frac{i}{r} b_{ll',mm'}^{kk'} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi_{l'm';\dot{l}'\dot{m}'}^{k'k'}}{\partial \tau} - \frac{i}{r} \mathbf{1}_{2l+1} \otimes b_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{k\dot{k}} \frac{\partial \psi_{l'm';\dot{l}'\dot{m}'}^{k'k'}}{\partial \tau} \right] \end{aligned}$$

$$\begin{aligned}
 & + (1 + i) c_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{kk';\dot{k}\dot{k}'} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial r} + \frac{1}{r} d_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{kk';\dot{k}\dot{k}'} \psi_{l'm';\dot{l}\dot{m}'}^{k'k'} + \frac{1}{r} e_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{kk';\dot{k}\dot{k}'} \psi_{l'm';\dot{l}\dot{m}'}^{k'k'} \\
 & + \left. \frac{2i}{r} \cot \theta^c a_{ll',mm'}^{kk'} \otimes \mathbf{1}_{2l+1} m' \psi_{l'm';\dot{l}\dot{m}'}^{k'k'} + \frac{2i}{r} \cot \theta^c \mathbf{1}_{2l+1} \otimes a_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{k\dot{k}'} m' \psi_{l'm';\dot{l}\dot{m}'}^{k'k'} \right] \\
 & + \kappa^c \psi_{lm;\dot{m}}^{kk} = 0, \\
 & \sum_{l',m',\dot{l}',\dot{m}',k',\dot{k}'} \left[\frac{1}{r^* \sin \theta^c} a_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk'} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial \varphi} - \frac{1}{r^* \sin \theta^c} \mathbf{1}_{2l+1} \otimes a_{ll',mm'}^{*kk'} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial \varphi} \right. \\
 & + \frac{i}{r^* \sin \theta^c} a_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk'} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial \epsilon} + \frac{i}{r^* \sin \theta^c} \mathbf{1}_{2l+1} \otimes a_{ll',mm'}^{*kk'} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial \epsilon} \\
 & - \frac{1}{r^*} b_{\dot{l}\dot{l}',2\dot{m}\dot{m}'}^{*kk'} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial \theta} + \frac{1}{r^*} \mathbf{1}_{2l+1} \otimes b_{ll',mm'}^{*kk'} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial \theta} - \\
 & - \frac{i}{r^*} b_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk'} \otimes \mathbf{1}_{2l+1} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial \tau} - \frac{i}{r^*} \mathbf{1}_{2l+1} \otimes b_{ll',mm'}^{*kk'} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial \tau} \\
 & + (1 - i) c_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk';\dot{k}\dot{k}'} \frac{\partial \psi_{l'm';\dot{l}\dot{m}'}^{k'k'}}{\partial r^*} + \frac{1}{r^*} d_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk';\dot{k}\dot{k}'} \psi_{l'm';\dot{l}\dot{m}'}^{k'k'} + \frac{1}{r^*} e_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk';\dot{k}\dot{k}'} \psi_{l'm';\dot{l}\dot{m}'}^{k'k'} \\
 & + \left. \frac{2i}{r^*} \cot \theta^c a_{\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk'} \otimes \mathbf{1}_{2l+1} m' \psi_{l'm';\dot{l}\dot{m}'}^{k'k'} + \frac{2i}{r^*} \cot \theta^c \mathbf{1}_{2l+1} \otimes a_{ll',mm'}^{*kk'} m' \psi_{l'm';\dot{l}\dot{m}'}^{k'k'} \right] \\
 & + \dot{\kappa}^c \psi_{lm;\dot{m}}^{kk} = 0, \tag{121}
 \end{aligned}$$

where the coefficients $a_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{kk';\dot{k}\dot{k}'}, \dots, e_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk';\dot{k}\dot{k}'}$ are defined by the formulae (74), (75), (72), (117), (77), (78), (76), (118), respectively.

With the view to separate the variables in (121), let us assume that

$$\begin{aligned}
 \psi_{lm;\dot{m}}^{kk} &= f_{lmk;\dot{m}\dot{k}}^{l_0\dot{l}_0}(r) \mathfrak{M}_{mn;\dot{m}\dot{n}}^{l_0\dot{l}_0}(\varphi, \epsilon, \theta, \tau, 0, 0), \\
 \psi_{lm;\dot{m}}^{k\dot{k}} &= f_{lmk;\dot{m}\dot{k}}^{*l_0\dot{l}_0}(r^*) \mathfrak{M}_{mn;\dot{m}\dot{n}}^*(l_0\dot{l}_0)(\varphi, \epsilon, \theta, \tau, 0, 0), \tag{122}
 \end{aligned}$$

where $l_0 \geq l, -l_0 \leq m, n \leq l_0$ and $\dot{l}_0 \geq \dot{l}, -\dot{l}_0 \leq \dot{m}, \dot{n} \leq \dot{l}_0$. Substituting the functions (122) into the system (121) and taking into account values of the coefficients $a_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{kk';\dot{k}\dot{k}'}, \dots, e_{ll',mm';\dot{l}\dot{l}',\dot{m}\dot{m}'}^{*kk';\dot{k}\dot{k}'}$, we collect together the terms with identical radial

functions. In the result, we obtain

$$\begin{aligned}
 & \sum_{k', k''} c_{l-1, l; i-1, i}^{kk'; k''k''} \left\{ \sqrt{(l^2 - m^2)(l^2 - \dot{m}^2)} \left[(1+i) \frac{\partial f_{l-1, m, k'; i-1, \dot{m}k''}^{l_0 l_0}}{\partial r} \right. \right. \\
 & \left. \left. + \frac{1}{r} (l - il + i + i\dot{l} - 2) f_{l-1, m, k'}^{l_0} \right] \mathfrak{M}_{mn; \dot{m}\dot{n}}^{l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) \right. \\
 & + \frac{1}{2r} \sqrt{(l+m)(l+m-1)(l^2 - \dot{m}^2)} f_{l-1, m-1, k'; i-1, \dot{m}k''}^{l_0 l_0} \left[-\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \varphi} \right. \\
 & \left. - \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \epsilon} + i \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \tau} - \frac{2i(m-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0} \right] \\
 & + \frac{1}{2r} \sqrt{(l-m)(l-m-1)(l^2 - \dot{m}^2)} f_{l-1, m+1, k'; i-1, \dot{m}k''}^{l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \varphi} \right. \\
 & \left. + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \epsilon} + i \frac{\partial \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \tau} \right. \\
 & \left. + \frac{2i(m+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0} \right] \\
 & + \frac{1}{2r} \sqrt{(l^2 - m^2)(l + \dot{m})(l + \dot{m} - 1)} f_{l-1, m, k'; i-1, \dot{m}-1, k''}^{l_0 l_0} \left[-\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}}{\partial \varphi} \right. \\
 & \left. + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}}{\partial \epsilon} + i \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}}{\partial \tau} + \frac{2i(\dot{m}-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0} \right] \\
 & + \frac{1}{2r} \sqrt{(l^2 - m^2)(l - \dot{m})(l - \dot{m} - 1)} f_{l-1, m, k'; i-1, \dot{m}+1, k''}^{l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}}{\partial \varphi} - \frac{i}{\sin \theta^c} \right. \\
 & \left. \times \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}}{\partial \epsilon} + i \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}}{\partial \tau} - \frac{2i(\dot{m}+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0} \right] \left. \right\} \\
 & + \sum_{k', k''} c_{l; i; j}^{kk'; k''k''} \left\{ \left[(1+i)m\dot{m} \frac{\partial f_{lmk'; j\dot{m}k''}^{l_0 l_0}}{\partial r} - \frac{2i}{r} m\dot{m} f_{lmk'; j\dot{m}k''}^{l_0 l_0} \right] \mathfrak{M}_{mn; \dot{m}\dot{n}}^{l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) \right. \\
 & \left. + \frac{1}{2r} \dot{m} \sqrt{(l+m)(l-m+1)} f_{l, m-1, k'; i\dot{m}k''}^{l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \varphi} + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \epsilon} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & -i \left[\frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \tau} + \frac{2i(m-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{l_0 l_0} \right] \\
 & + \frac{1}{2r} \dot{m} \sqrt{(l+m+1)(l-m)} f_{l,m+1,k';\dot{l}\dot{m}\dot{k}'}^{l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \varphi} + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \epsilon} \right. \\
 & \left. + i \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \tau} + \frac{2i(m+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0} \right] \Big\} \\
 & + \frac{1}{2r} m \sqrt{(\dot{l} + \dot{m})(\dot{l} - \dot{m} + 1)} f_{l\dot{m}k';\dot{j},\dot{m}-1,\dot{k}'}^{l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m\dot{n};\dot{m}-1,\dot{n}}^{l_0 l_0}}{\partial \varphi} - \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m\dot{n};\dot{m}-1,\dot{n}}^{l_0 l_0}}{\partial \epsilon} \right. \\
 & \left. - i \frac{\partial \mathfrak{M}_{m\dot{n};\dot{m}-1,\dot{n}}^{l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{m\dot{n};\dot{m}-1,\dot{n}}^{l_0 l_0}}{\partial \tau} - \frac{2i(\dot{m}-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m\dot{n};\dot{m}-1,\dot{n}}^{l_0 l_0} \right] \\
 & + \frac{1}{2r} m \sqrt{(\dot{l} + \dot{m} + 1)(\dot{l} - \dot{m})} f_{l\dot{m}k';\dot{j},\dot{m}+1,\dot{k}'}^{l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m\dot{n};\dot{m}+1,\dot{n}}^{l_0 l_0}}{\partial \varphi} - \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m\dot{n};\dot{m}+1,\dot{n}}^{l_0 l_0}}{\partial \epsilon} \right. \\
 & \left. + i \frac{\partial \mathfrak{M}_{m\dot{n};\dot{m}+1,\dot{n}}^{l_0 l_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{m\dot{n};\dot{m}+1,\dot{n}}^{l_0 l_0}}{\partial \tau} - \frac{2i(\dot{m}+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m\dot{n};\dot{m}+1,\dot{n}}^{l_0 l_0} \right] \Big\} \\
 & + \sum_{k',\dot{k}'} c^{kk';\dot{k}\dot{k}'}_{l+1,l;\dot{j}+1,\dot{i}} \left\{ \sqrt{((l+1)^2 - m^2)((\dot{l}+1)^2 - \dot{m}^2)} \left[(1+i) \frac{\partial f_{l+1,m,k';\dot{j}+1,\dot{m}\dot{k}'}^{l_0 l_0}}{\partial r} \right. \right. \\
 & \left. \left. + \frac{1}{r} (i\dot{l} - l - \dot{j} - i\dot{j} - 2) f_{l+1,m,k';\dot{j}+1,\dot{m}\dot{k}'}^{l_0 l_0} \right] \mathfrak{M}_{m\dot{n};\dot{m}\dot{n}}^{l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) \right. \\
 & \left. + \frac{1}{2r} \sqrt{(l-m+1)(l-m+2)((\dot{l}+1)^2 - \dot{m}^2)} f_{l+1,m-1,k';\dot{j}-1,\dot{m}\dot{k}'}^{l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \varphi} \right. \right. \\
 & \left. \left. + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \tau} + \frac{2i(m-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{l_0 l_0} \right] \right. \\
 & \left. + \frac{1}{2r} \sqrt{(l+m+1)(l+m+2)((\dot{l}+1)^2 - \dot{m}^2)} f_{l+1,m+1,k';\dot{j}+1,\dot{m}\dot{k}'}^{l_0 l_0} \left[-\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \varphi} \right. \right. \\
 & \left. \left. - \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0}}{\partial \tau} - \frac{2i(m+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{l_0 l_0} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2r} \sqrt{((l+1)^2 - m^2)(l - \dot{m} + 1)(l - \dot{m} + 2)} f_{l+1, mk'; j-1, \dot{m}-1, k'}^{l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}}{\partial \varphi} \right. \\
& - \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}}{\partial \tau} + \left. \frac{2i(\dot{m} - 1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0} \right] \\
& + \frac{1}{2r} \sqrt{((l+1)^2 - m^2)(l + \dot{m} + 1)(l + \dot{m} + 2)} f_{l+1, mk'; j+1, \dot{m}+1, k'}^{l_0 l_0} \left[-\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}}{\partial \varphi} \right. \\
& + \left. \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}}{\partial \tau} + \frac{2i(\dot{m} + 1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0} \right] \\
& + \kappa^c f_{lm; \dot{m}}^{l_0 l_0} \mathfrak{M}_{mn; \dot{m}\dot{n}}^{l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) = 0,
\end{aligned}$$

$$\begin{aligned}
& \sum_{k'} c_{l-1, l; i-1, i}^{*kk'; k'k'} \left\{ \sqrt{(l^2 - m^2)(l^2 - \dot{m}^2)} \left[(1-i) \frac{\partial f_{l-1, mk'; j-1, \dot{m}k'}^{*l_0 l_0}}{\partial r^*} \right. \right. \\
& - \left. \frac{1}{r^*} (l - il + l + il + 2) f_{l-1, mk'; j-1, \dot{m}k'}^{*l_0 l_0} \right] \mathfrak{M}_{mn; \dot{m}\dot{n}}^{*l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) \\
& + \frac{1}{2r^*} \sqrt{(l^2 - m^2)(l + \dot{m})(l + \dot{m} - 1)} f_{l-1, mk'; j-1, \dot{m}-1, k'}^{*l_0 l_0} \left[-\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0}}{\partial \varphi} \right. \\
& - \left. \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0}}{\partial \epsilon} + i \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0}}{\partial \tau} - \frac{2i(\dot{m} - 1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0} \right] \\
& + \frac{1}{2r^*} \sqrt{(l^2 - m^2)(l - \dot{m})(l - \dot{m} - 1)} f_{l-1, mk'; j-1, \dot{m}+1, k'}^{*l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0}}{\partial \varphi} \right. \\
& + \left. \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0}}{\partial \epsilon} + i \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0}}{\partial \tau} + \frac{2i(\dot{m} + 1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0} \right] \\
& + \frac{1}{2r^*} \sqrt{(l+m)(l+m-1)(l^2 - \dot{m}^2)} f_{l-1, m-1, k'; j-1, \dot{m}k'}^{*l_0 l_0} \left[-\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \varphi} \right.
\end{aligned}$$

$$\begin{aligned}
 & + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \epsilon} + i \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \tau} + \frac{2i(m-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0} \left. \right] \\
 & + \frac{1}{2r^*} \sqrt{(l-m)(l-m-1)(l^2 - \dot{m}^2)} \mathfrak{f}_{l-1,m+1,k';j-1,\dot{m}k'}^{*l_0\dot{l}_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \varphi} - \frac{i}{\sin \theta^c} \right. \\
 & \times \left. \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \epsilon} + i \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \tau} - \frac{2i(m+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0} \right] \\
 & + \sum_{k',k''} c_{ll;jj}^{*kk';kk'} \left\{ \left[(1-i)m\dot{m} \frac{\partial \mathfrak{f}_{lmk';j,\dot{m},k'}^{*l_0\dot{l}_0}}{\partial r^*} + \frac{2i}{r^*} m\dot{m} \mathfrak{f}_{lmk';j,\dot{m},k'}^{*l_0\dot{l}_0} \right] \mathfrak{M}_{mn;\dot{m},\dot{n}}^{*l_0\dot{l}_0}(\varphi, \epsilon, \theta, \tau, 0, 0) \right. \\
 & + \frac{1}{2r^*} m \sqrt{(j+\dot{m})(j-\dot{m}+1)} \mathfrak{f}_{lmk^p rime;j,\dot{m}-1,k'}^{*l_0\dot{l}_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn;\dot{m}-1,\dot{n}}^{*l_0\dot{l}_0}}{\partial \varphi} + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn;\dot{m}-1,\dot{n}}^{*l_0\dot{l}_0}}{\partial \epsilon} \right. \\
 & \left. - i \frac{\partial \mathfrak{M}_{mn;\dot{m}-1,\dot{n}}^{*l_0\dot{l}_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{mn;\dot{m}-1,\dot{n}}^{*l_0\dot{l}_0}}{\partial \tau} + \frac{2i(\dot{m}-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn;\dot{m}-1,\dot{n}}^{*l_0\dot{l}_0} \right] \\
 & + \frac{1}{2r^*} m \sqrt{(j+\dot{m}+1)(j-\dot{m})} \mathfrak{f}_{lmk';j,\dot{m}+1,k'}^{*l_0\dot{l}_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn;\dot{m}+1,\dot{n}}^{*l_0\dot{l}_0}}{\partial \varphi} + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn;\dot{m}+1,\dot{n}}^{*l_0\dot{l}_0}}{\partial \epsilon} \right. \\
 & \left. + i \frac{\partial \mathfrak{M}_{mn;\dot{m}+1,\dot{n}}^{*l_0\dot{l}_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{mn;\dot{m}+1,\dot{n}}^{*l_0\dot{l}_0}}{\partial \tau} + \frac{2i(\dot{m}+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn;\dot{m}+1,\dot{n}}^{*l_0\dot{l}_0} \right] \\
 & + \frac{1}{2r^*} \dot{m} \sqrt{(l+m)(l-m+1)} \mathfrak{f}_{l,m-1,k^p rime;j\dot{m}k'}^{*l_0\dot{l}_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \varphi} - \frac{i}{\sin \theta^c} \right. \\
 & \times \left. \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \tau} - \frac{2i(m-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m-1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0} \right] \\
 & + \frac{1}{2r^*} \dot{m} \sqrt{(l+m+1)(l-m)} \mathfrak{f}_{l,m+1,k';j\dot{m}k'}^{*l_0\dot{l}_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \varphi} - \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \epsilon} \right. \\
 & \left. + i \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0}}{\partial \tau} - \frac{2i(m+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m+1,n;\dot{m}\dot{n}}^{*l_0\dot{l}_0} \right] \left. \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{k', \bar{k}'}^{*kk'; \bar{k}\bar{k}'} C_{l+1, l; i+1, i} \left\{ \sqrt{((l+1)^2 - m^2)((j+1)^2 - \dot{m}^2)} \left[(1-i) \frac{\partial f_{l+1, mk'; i+1, \dot{m}, \bar{k}'}^{*l_0 l_0}}{\partial r^*} \right. \right. \\
& + \left. \frac{1}{r^*} (il - l - j - i\dot{l} - 2 + i2) f_{l+1, mk'; i+1, \dot{m}, \bar{k}'}^{*l_0 l_0} \right] \mathfrak{M}_{mn; \dot{m}\dot{n}}^{*l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) \\
& + \frac{1}{2r^*} \sqrt{((l+1)^2 - m^2)(j - \dot{m} + 1)(j - \dot{m} + 2)} f_{l+1, mk'; i+1, \dot{m}-1, \bar{k}'}^{*l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0}}{\partial \varphi} \right. \\
& + \left. \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0}}{\partial \tau} + \frac{2i(\dot{m} - 1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{*l_0 l_0} \right] \\
& + \frac{1}{2r^*} \sqrt{((l+1)^2 - m^2)(j + \dot{m} + 1)(j + \dot{m} + 2)} f_{l+1, mk'; i+1, \dot{m}+1, \bar{k}}^{*l_0 l_0} \left[- \frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0}}{\partial \varphi} \right. \\
& - \left. \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0}}{\partial \theta} + \frac{\partial \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0}}{\partial \tau} - \frac{2i(\dot{m} + 1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{*l_0 l_0} \right] \\
& + \frac{1}{2r^*} \sqrt{((l-m+1)(l-m+2)((j+1)^2 - \dot{m}^2)} f_{l+1, m-1, k'; i+1, \dot{m}, \bar{k}}^{*l_0 l_0} \left[\frac{1}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \varphi} \right. \\
& - \left. \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \tau} - \frac{2i(m-1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{*l_0 l_0} \right] \\
& + \frac{1}{2r^*} \sqrt{((l+m+1)(l+m+2)((j+1)^2 - \dot{m}^2)} f_{l+1, m+1, k'; i+1, \dot{m}, \bar{k}}^{*l_0 l_0} \left[- \frac{1}{\sin \theta^c} \right. \\
& \times \left. \frac{\partial \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \varphi} + \frac{i}{\sin \theta^c} \frac{\partial \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \epsilon} - i \frac{\partial \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{*l_0 l_0}}{\partial \tau} \right. \\
& \left. + \frac{2i(m+1) \cos \theta^c}{\sin \theta^c} \mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{*l_0 l_0} \right] \left. \right\} + \dot{k}^c f_{lm; \dot{l}\dot{m}}^{*l_0 l_0} \mathfrak{M}_{mn; \dot{m}\dot{n}}^{*l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) = 0. \quad (123)
\end{aligned}$$

The each equation of the system obtained contains five generalized hyperspherical functions $\mathfrak{M}_{mn; \dot{m}\dot{n}}^{l_0 l_0}$, $\mathfrak{M}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}$, $\mathfrak{M}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0}$, $\mathfrak{M}_{mn; \dot{m}-1, \dot{n}}^{l_0 l_0}$, $\mathfrak{M}_{mn; \dot{m}+1, \dot{n}}^{l_0 l_0}$ and their conjugate. Let us recall that $\mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{-n(\epsilon+i\varphi) - \dot{n}(\epsilon-i\varphi)} \mathfrak{Z}_{m\pm 1, n; \dot{m}\dot{n}}^{l_0 l_0}(\theta, \tau)$, $\mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{-n(\epsilon+i\varphi) - \dot{n}(\epsilon-i\varphi)} \mathfrak{Z}_{mn; \dot{m}\pm 1, \dot{n}}^{l_0 l_0}(\theta, \tau)$ and $\mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{*l_0 l_0}(\varphi, \epsilon, \theta, \tau,$

$$0, 0) = e^{-n(\epsilon-i\varphi)-\dot{n}(\epsilon-i\varphi)} \mathfrak{Z}_{m\pm 1, n; \dot{m}\dot{n}}^{* l_0 l_0}(\theta, \tau), \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{* l_0 l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{-n(\epsilon-i\varphi)-\dot{n}(\epsilon-i\varphi)} \mathfrak{Z}_{mn; \dot{m}\pm 1, \dot{n}}^{* l_0 l_0}(\theta, \tau).$$

Therefore,

$$\begin{aligned} \frac{\partial \mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \varphi} &= (-in + i\dot{n}) \mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{l_0 l_0}, & \frac{\partial \mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \epsilon} &= -(n + \dot{n}) \mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{l_0 l_0}, \\ \frac{\partial \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{l_0 l_0}}{\partial \varphi} &= (-in + i\dot{n}) \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{l_0 l_0}, & \frac{\partial \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{l_0 l_0}}{\partial \epsilon} &= -(n + \dot{n}) \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{l_0 l_0}, \\ \frac{\partial \mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{* l_0 l_0}}{\partial \varphi} &= (in - i\dot{n}) \mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{* l_0 l_0}, & \frac{\partial \mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{* l_0 l_0}}{\partial \epsilon} &= -(n + \dot{n}) \mathfrak{M}_{m\pm 1, n; \dot{m}\dot{n}}^{* l_0 l_0}, \\ \frac{\partial \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{* l_0 l_0}}{\partial \varphi} &= (in - i\dot{n}) \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{* l_0 l_0}, & \frac{\partial \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{* l_0 l_0}}{\partial \epsilon} &= -(n + \dot{n}) \mathfrak{M}_{mn; \dot{m}\pm 1, \dot{n}}^{* l_0 l_0}. \end{aligned}$$

We apply now the recurrence relations (36)–(39), (42)–(45) to square brackets containing the hyperspherical functions. For example, in virtue of (37) the second bracket in (123) can be written as

$$\begin{aligned} e^{-n(\epsilon+i\varphi)-\dot{n}(\epsilon-i\varphi)} &\left[i \frac{\partial \mathfrak{Z}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{Z}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \tau} + \frac{2i(n - (m - 1) \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{m-1, n; \dot{m}\dot{n}}^{l_0 l_0} \right] \\ &= 2\sqrt{(l_0 + m)(l_0 - m + 1)} \mathfrak{M}_{mn; \dot{m}\dot{n}}^{l_0 l_0}. \end{aligned} \tag{124}$$

Further, in virtue of (36) for the third bracket we have

$$\begin{aligned} e^{-n(\epsilon+i\varphi)-\dot{n}(\epsilon-i\varphi)} &\left[i \frac{\partial \mathfrak{Z}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \theta} - \frac{\partial \mathfrak{Z}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0}}{\partial \tau} - \frac{2i(n - (m + 1) \cos \theta^c)}{\sin \theta^c} \mathfrak{Z}_{m+1, n; \dot{m}\dot{n}}^{l_0 l_0} \right] \\ &= 2\sqrt{(l_0 + m + 1)(l_0 - m)} \mathfrak{M}_{mn; \dot{m}\dot{n}}^{l_0 l_0}. \end{aligned} \tag{125}$$

and so on. In doing so, we replace all the square brackets in the system (123) via the relations of the type (124) and (125) and cancel all the equations by $\mathfrak{M}_{mn; \dot{m}\dot{n}}^{l_0 l_0}$ ($\mathfrak{M}_{mn; \dot{m}\dot{n}}^{* l_0 l_0}$). In the result, we see that the relativistically invariant system is reduced to a system of ordinary differential equations,

$$\begin{aligned} \sum_{k', \dot{k}'} c_{l-1, l; j-1, j}^{kk'; \dot{k}\dot{k}'} &\left[(1 + i) \sqrt{(l^2 - m^2)(\dot{l}^2 - \dot{m}^2)} \frac{d \mathbf{f}_{l-1, m k'; j-1, \dot{m} \dot{k}'}^{l_0 l_0}}{dr} \right. \\ &\left. + \frac{1}{r} (l - il + j + i\dot{j} - 2i) \sqrt{(l^2 - m^2)(\dot{l}^2 - \dot{m}^2)} \mathbf{f}_{l-1, m k'; j-1, \dot{m} \dot{k}'}^{l_0 l_0} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{r} \sqrt{(l+m)(l+m-1)(l^2 - \dot{m}^2)} \sqrt{(l_0+m)(l_0-m+1)} \mathbf{f}_{l-1, m-1, k'; j-1, \dot{m} \dot{k}'}^{l_0 l_0} \\
& + \frac{1}{r} \sqrt{(l-m)(l-m-1)(l^2 - \dot{m}^2)} \sqrt{(l_0+m+1)(l_0-m)} \mathbf{f}_{l-1, m+1, k'; j-1, \dot{m} \dot{k}'}^{l_0 l_0} \\
& + \frac{1}{r} \sqrt{(l^2 - m^2)(\dot{l} + \dot{m})(\dot{l} + \dot{m} - 1)} \sqrt{(\dot{l}_0 + \dot{m})(\dot{l}_0 - \dot{m} + 1)} \mathbf{f}_{l-1, m, k'; j-1, \dot{m}-1, \dot{k}'}^{l_0 l_0} \\
& + \frac{1}{r} \sqrt{(l^2 - m^2)(\dot{l} - \dot{m})(\dot{l} - \dot{m} - 1)} \sqrt{(\dot{l}_0 + \dot{m} + 1)(\dot{l}_0 - \dot{m})} \mathbf{f}_{l-1, m, k'; j-1, \dot{m}+1, \dot{k}'}^{l_0 l_0} \Big] \\
& + \sum_{k', \dot{k}'} c_{ll; \dot{i}\dot{i}}^{kk'; \dot{k}\dot{k}'} \left[(1+i) m \dot{m} \frac{d \mathbf{f}_{lmk'; \dot{i}\dot{m}\dot{k}}^{l_0 l_0}}{dr} - \frac{2i}{r} m \dot{m} \mathbf{f}_{lmk'; \dot{i}\dot{m}\dot{k}}^{l_0 l_0} \right. \\
& - \frac{1}{r} \dot{m} \sqrt{(l+m)(l-m+1)} \sqrt{(l_0+m)(l_0-m+1)} \mathbf{f}_{l, m-1, k'; \dot{i}\dot{m}\dot{k}'}^{l_0 l_0} \\
& + \frac{1}{r} \dot{m} \sqrt{(l+m+1)(l-m)} \sqrt{(l_0+m+1)(l_0-m)} \mathbf{f}_{l, m+1, k'; \dot{i}\dot{m}\dot{k}'}^{l_0 l_0} \\
& - \frac{1}{r} m \sqrt{(\dot{l} + \dot{m})(\dot{l} - \dot{m} + 1)} \sqrt{(\dot{l}_0 + \dot{m})(\dot{l}_0 - \dot{m} + 1)} \mathbf{f}_{lmk'; \dot{i}, \dot{m}-1, \dot{k}'}^{l_0 l_0} \\
& \left. + \frac{1}{r} m \sqrt{(\dot{l} + \dot{m} + 1)(\dot{l} - \dot{m})} \sqrt{(\dot{l}_0 + \dot{m} + 1)(\dot{l}_0 - \dot{m})} \mathbf{f}_{lmk'; \dot{i}, \dot{m}+1, \dot{k}'}^{l_0 l_0} \right] \\
& + \sum_{k', \dot{k}'} c_{l+1, l; \dot{i}+1, \dot{i}}^{kk'; \dot{k}\dot{k}'} \left[(1+i) \sqrt{((l+1)^2 - m^2)((\dot{l}+1)^2 - \dot{m}^2)} \frac{d \mathbf{f}_{l+1, m, k'; \dot{i}+1, \dot{m}\dot{k}'}^{l_0 l_0}}{dr} \right. \\
& + \frac{1}{r} (il - l - \dot{l} - i\dot{l} - 2) \sqrt{((l+1)^2 - m^2)((\dot{l}+1)^2 - \dot{m}^2)} \mathbf{f}_{l+1, m, k'; \dot{i}+1, \dot{m}\dot{k}'}^{l_0 l_0} \\
& - \frac{1}{r} \sqrt{(l-m+1)(l-m+2)((\dot{l}+1)^2 - \dot{m}^2)} \sqrt{(l_0+m)(l_0-m+1)} \mathbf{f}_{l+1, m-1, k'; \dot{i}-1, \dot{m}\dot{k}'}^{l_0 l_0} \\
& - \frac{1}{r} \sqrt{(l+m+1)(l+m+2)((\dot{l}+1)^2 - \dot{m}^2)} \sqrt{(l_0+m+1)(l_0-m)} \mathbf{f}_{l+1, m+1, k'; \dot{i}+1, \dot{m}\dot{k}'}^{l_0 l_0} \\
& - \frac{1}{r} \sqrt{((l+1)^2 - m^2)(\dot{l} - \dot{m} + 1)(\dot{l} - \dot{m} + 2)} \sqrt{(\dot{l}_0 + \dot{m})(\dot{l}_0 - \dot{m})} \mathbf{f}_{l+1, m, k'; \dot{i}-1, \dot{m}-1, \dot{k}'}^{l_0 l_0} \\
& - \frac{1}{r} \sqrt{((l+1)^2 - m^2)(\dot{l} + \dot{m} + 1)(\dot{l} + \dot{m} + 2)} \sqrt{(\dot{l}_0 + \dot{m} + 1)(\dot{l}_0 - \dot{m})} \mathbf{f}_{l+1, m, k'; \dot{i}+1, \dot{m}+1, \dot{k}'}^{l_0 l_0} \Big] \\
& + \kappa^c \mathbf{f}_{lmk}^{l_0}(r) = 0,
\end{aligned}$$

$$\begin{aligned}
 & \sum_{k',\dot{k}'} c_{l-1,l;i,j-1,i}^{*kk';\dot{k}\dot{k}'} \left[(1-i)\sqrt{(l^2-m^2)(\dot{l}^2-\dot{m}^2)} \frac{d}{dr^*} f_{l-1,mk';i-1,\dot{m}\dot{k}'}^{l\dot{l}o} \right. \\
 & + \frac{1}{r^*} (l-il+i+i\dot{j}+2i)\sqrt{(l^2-m^2)(\dot{l}^2-\dot{m}^2)} f_{l-1,mk';i-1,\dot{m}\dot{k}'}^{l\dot{l}o} \\
 & + \frac{1}{r^*} \sqrt{(l^2-m^2)(\dot{l}+\dot{m})(\dot{l}+\dot{m}-1)} \sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)} f_{l-1,mk';i-1,\dot{m}-1,\dot{k}'}^{l\dot{l}o} \\
 & + \frac{1}{r^*} \sqrt{(l^2-m^2)(\dot{l}-\dot{m})(\dot{l}-\dot{m}-1)} \sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})} f_{l-1,mk';i-1,\dot{m}+1,\dot{k}'}^{l\dot{l}o} \\
 & + \frac{1}{r^*} \sqrt{(l+m)(l+m-1)(\dot{l}^2-\dot{m}^2)} \sqrt{(\dot{l}_0+m)(\dot{l}_0-m+1)} f_{l-1,m-1,k';i-1,\dot{m}\dot{k}'}^{l\dot{l}o} \\
 & \left. + \frac{1}{r^*} \sqrt{(l-m)(l-m-1)(\dot{l}^2-\dot{m}^2)} \sqrt{(\dot{l}_0+m+1)(\dot{l}_0-m)} f_{l-1,m+1,k';i-1,\dot{m}\dot{k}'}^{l\dot{l}o} \right] \\
 & + \sum_{k',\dot{k}'} c_{ll;i\dot{i}}^{*kk';\dot{k}\dot{k}'} \left[(1-i)m\dot{m} \frac{d}{dr^*} f_{lmk';i\dot{m}\dot{k}'}^{l\dot{l}o} + \frac{2i}{r^*} m\dot{m} f_{lmk';i\dot{m}\dot{k}'}^{l\dot{l}o} \right. \\
 & - \frac{1}{r^*} m \sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)} \sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m})} f_{lmk';i,\dot{m}-1,\dot{k}'}^{l\dot{l}o} \\
 & + \frac{1}{r^*} m \sqrt{(\dot{l}+\dot{m}+1)(\dot{l}-\dot{m})} \sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})} f_{lmk';i,\dot{m}+1,\dot{k}'}^{l\dot{l}o} \\
 & - \frac{1}{r^*} \dot{m} \sqrt{(l+m)(l-m+1)} \sqrt{(\dot{l}_0+m)(\dot{l}_0-m+1)} f_{l,m-1,k';i\dot{m}\dot{k}'}^{l\dot{l}o} \\
 & \left. + \frac{1}{r^*} \dot{m} \sqrt{(l+m+1)(l-m)} \sqrt{(\dot{l}_0+m+1)(\dot{l}_0-m)} f_{l,m+1,k';i\dot{m}\dot{k}'}^{l\dot{l}o} \right] \\
 & + \sum_{k',\dot{k}'} c_{l+1,l;i,j+1,i}^{*kk';\dot{k}\dot{k}'} \left[(1-i)\sqrt{((l+1)^2-m^2)((\dot{l}+1)^2-\dot{m}^2)} \frac{d}{dr^*} f_{l+1,mk';i+1,\dot{m}\dot{k}'}^{l\dot{l}o} \right. \\
 & + \frac{1}{r^*} (il-l-j-i\dot{j}-2+2i)\sqrt{((l+1)^2-m^2)((\dot{l}+1)^2-\dot{m}^2)} f_{l+1,mk';i+1,\dot{m}\dot{k}'}^{l\dot{l}o} \\
 & + \frac{1}{r^*} \sqrt{((l+1)^2-m^2)(\dot{l}-\dot{m}+1)(\dot{l}-\dot{m}+2)} \sqrt{(\dot{l}_0+\dot{m})(\dot{l}_0-\dot{m}+1)} f_{l+1,mk';i+1,\dot{m}-1,\dot{k}'}^{l\dot{l}o} \\
 & - \frac{1}{r^*} \sqrt{((l+1)^2-m^2)(\dot{l}+\dot{m}+1)(\dot{l}+\dot{m}+2)} \sqrt{(\dot{l}_0+\dot{m}+1)(\dot{l}_0-\dot{m})} f_{l+1,mk';i+1,\dot{m}+1,\dot{k}'}^{l\dot{l}o} \\
 & \left. \right]
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{r^*} \sqrt{(l-m+1)(l-m+2)((j+1)^2 - \dot{m}^2)} \sqrt{(l_0+m)(l_0-m+1)} f_{l+1, m-1, k'; j+1, \dot{m}k'}^{*l_0 l_0} \\
& -\frac{1}{r^*} \sqrt{(l+m+1)(l+m+2)((j+1)^2 - \dot{m}^2)} \sqrt{(l_0+m+1)(l_0-m)} f_{l+1, m+1, k'; j+1, \dot{m}k'}^{*l_0 l_0} \Big] \\
& + k^c f_{lm; \dot{m}}^{*l_0 l_0} = 0.
\end{aligned} \tag{126}$$

Substituting solutions of this system into the series (82) and (83), we obtain a solution of the boundary value problem.

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